Negative and fractional powers

In many calculations you will need to use negative and fractional powers. These are explained on this leaflet.

Negative powers

Negative powers are interpreted as follows:

\[ a^{-m} = \frac{1}{a^m} \quad \text{or equivalently} \quad a^m = \frac{1}{a^{-m}} \]

Examples

\[ 3^{-2} = \frac{1}{3^2}, \quad \frac{1}{5^{-2}} = 5^2, \quad x^{-1} = \frac{1}{x}, \quad x^{-2} = \frac{1}{x^2}, \quad 2^{-5} = \frac{1}{2^5} \]

Exercises

1. Write the following using only positive powers:
   (a) \( \frac{1}{x^{-6}} \), (b) \( x^{-12} \), (c) \( t^{-3} \), (d) \( \frac{1}{4^{-3}} \), (e) \( 5^{-17} \).

2. Without using a calculator evaluate (a) \( 2^{-3} \), (b) \( 3^{-2} \), (c) \( \frac{1}{4^{-2}} \), (d) \( \frac{1}{2^{-5}} \), (e) \( \frac{1}{4^{-3}} \).

Fractional powers

To understand fractional powers you first need to have an understanding of roots, and in particular square roots and cube roots.

When a number is raised to a fractional power this is interpreted as follows:

\[ a^{1/n} = \sqrt[n]{a} \]

So,

\[ a^{1/2} \quad \text{is a square root of } a \]
\[ a^{1/3} \quad \text{is the cube root of } a \]
\[ a^{1/4} \quad \text{is a fourth root of } a \]

Examples

\[ 3^{1/2} = \sqrt{3}, \quad 27^{1/3} = \sqrt[3]{27} \quad \text{or} \quad 3, \quad 32^{1/5} = \sqrt[5]{32} = 2, \]
\[ 64^{1/3} = \sqrt[3]{64} = 4, \quad 81^{1/4} = \sqrt[4]{81} = 3 \]
Fractional powers are useful when we need to calculate roots using a scientific calculator. For example to find $\sqrt[7]{38}$ we rewrite this as $38^{1/7}$ which can be evaluated using a scientific calculator. You may need to check your calculator manual to find the precise way of doing this, probably with the buttons $x^y$ or $x^{1/y}$.

Check that you are using your calculator correctly by confirming that

$$38^{1/7} = 1.6814 \quad (4 \text{ dp})$$

More generally we can write:

$$a^{m/n} = \sqrt[n]{a^m} \text{ or equivalently } \left(\sqrt[n]{a}\right)^m$$

**Examples**

$$8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4, \quad \text{and} \quad 32^{3/5} = \left(\sqrt[5]{32}\right)^3 = 2^3 = 8$$

Alternatively,

$$8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4, \quad \text{and} \quad 32^{3/5} = \sqrt[5]{32^3} = \sqrt[5]{32768} = 8$$

**Exercises**

3. Use a calculator to find, correct to 4 decimal places, a) $\sqrt[5]{96}$, b) $\sqrt[3]{32}$.

4. Without using a calculator, evaluate a) $4^{3/2}$, b) $27^{2/3}$.

5. Use the rule $a^n = a^{n-m}$ with $n = 0$ to prove that $a^{-m} = \frac{1}{a^m}$.

6. Each of the following expressions can be written as $a^n$. Determine $n$ in each case:

   (a) $\frac{1}{a^5}$  \quad (b) $\sqrt{a} \times \frac{1}{a^2}$  \quad (c) 1  \quad (d) $\frac{1}{\sqrt{a}}$.

**Answers**

1. (a) $x^6$,  \quad (b) $\frac{1}{x^{12}}$,  \quad (c) $\frac{1}{y^3}$,  \quad (d) $4^3$,  \quad (e) $\frac{1}{5^{17}}$.

2. (a) $a^{-3} = \frac{1}{a^3} = \frac{1}{8}$,  \quad (b) $\frac{1}{9}$,  \quad (c) 16,  \quad (d) 32,  \quad (e) 64.

3. a) 2.4915,  \quad b) 2.3784.  \quad 4. a) $4^{3/2} = 8$,  \quad b) $27^{2/3} = 9$.

6. a) $-5$  \quad b) $-\frac{3}{2}$  \quad (c) 0  \quad (d) $-\frac{1}{2}$.