The addition formulae

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There are six so-called **addition formulae** often needed in the solution of trigonometric problems. In this unit we start with one and derive a second from that. Then we take another one as given, and derive a second one from that. Finally we use these four to help us derive the final two. This exercise will improve your familiarity and confidence in working with the addition formulae. The proofs of the formulae are left as structured exercises for you to complete.

In order to master the techniques explained here it is vital that you undertake the practice exercises provided.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- work with the six addition formulae

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1. Introduction

There are six so-called **addition formulae** often needed in the solution of trigonometric problems. In this unit we start with one and derive a second from that. Then we take another one as given, and derive a second one from that. And then we are going to use these four to help us derive the final two. This exercise will improve your familiarity and confidence in working with the addition formulae.

2. The first two addition formulae: $\sin(A \pm B)$

The formula we are going to start with is

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

This is called an addition formula because of the sum $A + B$ appearing the formula. Note that it enables us to express the sine of the sum of two angles in terms of the sines and cosines of the individual angles.

Key Point

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

We now want to look at $\sin(A - B)$. We can obtain a formula for $\sin(A - B)$ by replacing the B in the formula for $\sin(A + B)$ by $-B$. Then

$$\sin(A - B) = \sin A \cos(-B) + \cos A \sin(-B)$$

We now use the following important facts: $\cos(-B) = \cos B$, but $\sin(-B) = -\sin B$. Then

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

This is the second of our addition formulae.

Key Point

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

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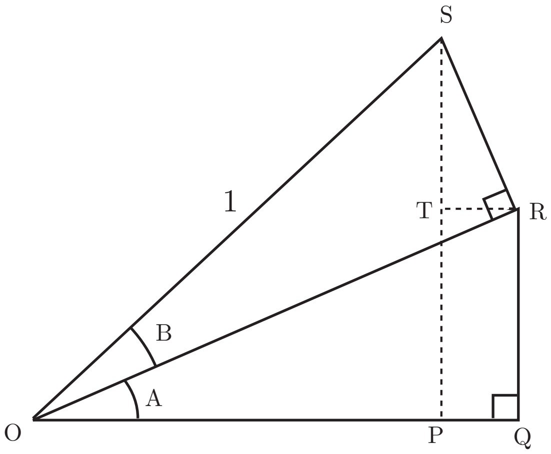
Exercise 1



1. By using right angled triangle OSR, in which the length of OS equals 1, determine the length of OR in terms of angle B.
2. By using the answer of part 1 and right angled triangle ORQ determine the length of QR in terms of angles A and B.
3. By using the answer of part 2 determine the length of PT.
4. What is ∠ TRO?
5. What is ∠ TRS?
6. What is ∠ RST?
7. By using right angled triangle OSR determine the length of RS.
8. By using the answer of part 7 and right angled triangle RST determine the length of TS.
9. By using the answers of parts 3 and 8 determine the length of PS.
10. By using the answer of part 9 and right angled triangle OSP determine $\sin(A + B)$.

3. The second two addition formulae: $\cos(A \pm B)$

This time, the addition formula we are going to start with is

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

Key Point

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

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We want to use this to derive another formula for $\cos(A-B)$. To do this, as before, we replace B with $-B$. This gives

$$\cos(A-B) = \cos A\cos(-B) - \sin A\sin(-B)$$

But $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$, and so

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

Key Point

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

So we've now got four addition formulae. We will summarise them all here:

Key Point

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

Exercise 2

Refer back to the figure in Exercise 1. Use a similar strategy to that of exercise 1 to determine lengths PQ (=TR), OQ and hence OP. From this determine $\cos(A+B)$.

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4. Deriving the two formulae for $\tan(A \pm B)$

From the four formulae we have seen already, it is possible to derive two more formulae. We can derive a formula for $\tan(A+B)$ from the earlier formulae by noting that

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

Then,

$$\begin{aligned}\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\ &= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}\end{aligned}$$

This result gives $\tan(A+B)$ in terms of sines and cosines. We now look at how we can write it directly in terms of $\tan A$ and $\tan B$. We do this by dividing every term, both top and bottom, on the right-hand side by $\cos A\cos B$. This produces

$$\tan(A+B) = \frac{\dfrac{\sin A\ \cos B}{\cos A\ \cos B} + \dfrac{\cos A\ \ \sin B}{\cos A\ \ \cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

Cancelling common factors where possible produces

$$\tan(A+B) = \frac{\dfrac{\sin A\ \cancel{\cos B}}{\cos A\ \cancel{\cos B}} + \dfrac{\cancel{\cos A}\ \ \sin B}{\cancel{\cos A}\ \ \cos B}}{\dfrac{\cancel{\cos A\cos B}}{\cancel{\cos A\cos B}} - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

so that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

We can do the same with $\tan(A-B)$ which would produce

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Key Point

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

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5. Examples of the use of the formulae

Let's have a look at some fairly typical examples of when we need to use the addition formulae.

Example

Suppose we know that $\sin A = \frac{3}{5}$ and that $\cos B = \frac{5}{13}$ where A and B are acute angles. Suppose we want to use this information to find $\sin(A+B)$ and $\cos(A-B)$. Before we can use the addition formulae we need to know expressions for $\cos A$ and $\sin B$. We can find these by referring to the right-angled triangle in Figure 1.

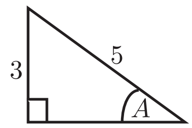

Figure 1. A right-angled triangle constructed from the given information: $\sin A = \frac{3}{5}$

Using Pythagoras' theorem we can deduce that the length of the third side is 4 as shown in Figure 2. Hence $\cos A = \frac{4}{5}$.

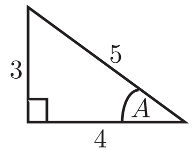

Figure 2. From the right-angled triangle, $\cos A = \frac{4}{5}$

Similarly, given that $\cos B = \frac{5}{13}$, then by reference to the triangle in Figure 3 and by using Pythagoras' theorem we can deduce that $\sin B = \frac{12}{13}$.

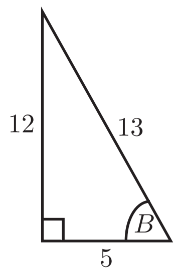

Figure 3. From the triangle $\sin B = \frac{12}{13}$.

We are now in a position to use the addition formulae:

$$\begin{aligned}\sin(A+B) &= \sin A \cos B + \cos A \sin B \\ &= \frac{3}{5} \times \frac{5}{13} + \frac{4}{5} \times \frac{12}{13} = \frac{15}{65} + \frac{48}{65} = \frac{63}{65}\end{aligned}$$

$$\begin{aligned}\cos(A-B) &= \cos A \cos B + \sin A \sin B \\ &= \frac{4}{5} \times \frac{5}{13} + \frac{3}{5} \times \frac{12}{13} = \frac{20}{65} + \frac{36}{65} = \frac{56}{65}\end{aligned}$$

This is one way in which the formulae can be used.

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Example

Suppose we are asked to find an expression for $\sin 75°$, not by using a calculator but by using a combination of other known quantities. Note that we can rewrite $\sin 75°$ as $\sin(45° + 30°)$ and then use an addition formula. We have specifically chosen the values $45°$ and $30°$ because of the standard results that $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$, $\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$. Then

$$\begin{aligned}
\sin(A+B) &= \sin A \cos B + \cos A \sin B \\
\sin(45° + 30°) &= \sin 45° \cos 30° + \cos 45° \sin 30° \\
&= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} \\
&= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\
&= \frac{\sqrt{3}+1}{2\sqrt{2}}
\end{aligned}$$

Example

Suppose we wish to find an expression for $\tan 15°$ using known results. Note that $15° = 60° - 45°$ and also that $\tan 60° = \sqrt{3}$ and $\tan 45° = 1$.

$$\begin{aligned}
\tan 15° &= \tan(60° - 45°) \\
&= \frac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°} \\
&= \frac{\sqrt{3}-1}{1+\sqrt{3}\times 1} \\
&= \frac{\sqrt{3}-1}{\sqrt{3}+1}
\end{aligned}$$

It would be more usual to tidy this result up to avoid leaving a root in the denominator. This can be done by multiplying top and bottom by the same quantity, as follows:

$$\begin{aligned}
\frac{\sqrt{3}-1}{\sqrt{3}+1} &= \frac{(\sqrt{3}-1)}{\sqrt{3}+1} \times \frac{(\sqrt{3}-1)}{\sqrt{3}-1} \\
&= \frac{3-\sqrt{3}-\sqrt{3}+1}{3-1} \\
&= \frac{4-2\sqrt{3}}{2} \\
&= 2-\sqrt{3}
\end{aligned}$$

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Example

In this Example we use an addition formula to simplify an expression.

Suppose we have $\sin(90^\circ + A)$ and we want to write it in a different form.
We can use the first addition formula as follows:

$$\begin{aligned}\sin(90^\circ + A) &= \sin 90^\circ \cos A + \cos 90^\circ \sin A \\ &= \cos A\end{aligned}$$

since $\sin 90^\circ = 1$ and $\cos 90^\circ = 0$. So $\sin(90^\circ + A)$ can be written in the simpler form $\cos A$.

Example

Suppose we wish to simplify $\cos(180^\circ - A)$.

$$\begin{aligned}\cos(180^\circ - A) &= \cos 180^\circ \cos A + \sin 180^\circ \sin A \\ &= -\cos A\end{aligned}$$

since $\cos 180^\circ = -1$ and $\sin 180^\circ = 0$.

So we can see that these addition formulae help us to simplify quite complicated expressions.

Exercise 3

1. Verify each of the three addition formulae (i.e. for $\sin(A+B)$, $\cos(A+B)$, $\tan(A+B)$) for the cases:
   a) $A = 60^\circ$, $B = 30^\circ$ and b) $A = 45^\circ$, $B = 45^\circ$.

2. Verify each of the three subtraction formulae (i.e. for $\sin(A-B)$, $\cos(A-B)$, $\tan(A-B)$) for the cases:
   a) $A = 90^\circ$, $B = 60^\circ$ and b) $A = 90^\circ$, $B = 45^\circ$.

3. Angles A, B and C are acute angles such that $\sin A = 0.1$, $\cos B = 0.4$, $\sin C = 0.7$. Without finding angles A, B and C, use the addition formulae to calculate, to 2 decimal places,
   a) $\sin(A+B)$ b) $\cos(B-C)$ c) $\sin(C-A)$
   d) $\cos(A+C)$ e) $\tan(B-A)$ f) $\tan(C+B)$
   [Hint: Work to 4 decimal places when finding $\cos A$, $\tan A$, etc.]

4. By finding the angles A, B and C in question 3 verify your answers.

Answers

Exercise 1

1. $\cos B$ 2. $\sin A \cos B$ 3. $\sin A \cos B$ 4. A 5. $90^{\text{o}} - A$ 6. A 7. $\sin B$
8. $\cos A \sin B$ 9. $\sin A \cos B + \cos A \sin B$ 10. $\sin A \cos B + \cos A \sin B$

Exercise 2

PQ = TR = RS $\sin A = \sin A \sin B$
OQ = OR $\cos A = \cos A \cos B$
OP = OQ - PQ = $\cos A \cos B - \sin A \sin B$
$\cos(A+B) = \mathrm{OP} = \cos A \cos B - \sin A \sin B$

Exercise 3

3. a) 0.95 b) 0.93 c) 0.63 d) 0.64 e) 1.78 f) -2.63

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