# Arithmetic and geometric progressions

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This unit introduces sequences and series, and gives some simple examples of each. It also explores particular types of sequence known as arithmetic progressions (APs) and geometric progressions (GPs), and the corresponding series.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- recognise the difference between a sequence and a series;
- recognise an arithmetic progression;
- find the $n$-th term of an arithmetic progression;
- find the sum of an arithmetic series;
- recognise a geometric progression;
- find the $n$-th term of a geometric progression;
- find the sum of a geometric series;
- find the sum to infinity of a geometric series with common ratio $|r| < 1$.

## Contents

# 1. Sequences

What is a sequence? It is a set of numbers which are written in some particular order. For example, take the numbers

$$1, \;\; 3, \;\; 5, \;\; 7, \;\; 9, \;\; \ldots .$$

Here, we seem to have a rule. We have a sequence of odd numbers. To put this another way, we start with the number 1, which is an odd number, and then each successive number is obtained by adding 2 to give the next odd number.

Here is another sequence:

$$1, \;\; 4, \;\; 9, \;\; 16, \;\; 25, \;\; \ldots .$$

This is the sequence of square numbers. And this sequence,

$$1, \;\; -1, \;\; 1, \;\; -1, \;\; 1, \;\; -1, \;\; \ldots ,$$

is a sequence of numbers alternating between 1 and $-1$. In each case, the dots written at the end indicate that we must consider the sequence as an infinite sequence, so that it goes on for ever.

On the other hand, we can also have finite sequences. The numbers

$$1, \;\; 3, \;\; 5, \;\; 9$$

form a finite sequence containing just four numbers. The numbers

$$1, \;\; 4, \;\; 9, \;\; 16$$

also form a finite sequence. And so do these, the numbers

$$1, \;\; 2, \;\; 3, \;\; 4, \;\; 5, \;\; 6, \;\; \ldots, \;\; n .$$

These are the numbers we use for counting, and we have included $n$ of them. Here, the dots indicate that we have not written all the numbers down explicitly. The $n$ after the dots tells us that this is a finite sequence, and that the last number is $n$.

Here is a sequence that you might recognise:

$$1, \;\; 1, \;\; 2, \;\; 3, \;\; 5, \;\; 8, \;\; \ldots .$$

This is an infinite sequence where each term (from the third term onwards) is obtained by adding together the two previous terms. This is called the Fibonacci sequence.

We often use an algebraic notation for sequences. We might call the first term in a sequence $u_1$, the second term $u_2$, and so on. With this same notation, we would write $u_n$ to represent the $n$-th term in the sequence. So

$$u_1, \;\; u_2, \;\; u_3, \;\; \ldots, \;\; u_n$$

would represent a finite sequence containing $n$ terms. As another example, we could use this notation to represent the rule for the Fibonacci sequence. We would write

$$u_n = u_{n-1} + u_{n-2}$$

to say that each term was the sum of the two preceding terms.

 

### Key Point

A sequence is a set of numbers written in a particular order. We sometimes write $u_1$ for the first term of the sequence, $u_2$ for the second term, and so on. We write the $n$-th term as $u_n$.

**Exercise 1**

(a) A sequence is given by the formula $u_n = 3n + 5$, for $n = 1, 2, 3, \ldots$. Write down the first five terms of this sequence.

(b) A sequence is given by $u_n = 1/n^2$, for $n = 1, 2, 3, \ldots$. Write down the first four terms of this sequence. What is the 10th term?

(c) Write down the first eight terms of the Fibonacci sequence defined by $u_n = u_{n-1} + u_{n-2}$, when $u_1 = 1$, and $u_2 = 1$.

(d) Write down the first five terms of the sequence given by $u_n = (-1)^{n+1}/n$.

## 2. Series

A *series* is something we obtain from a sequence by adding all the terms together.

For example, suppose we have the sequence

$$u_1, \quad u_2, \quad u_3, \quad \ldots, \quad u_n\,.$$

The series we obtain from this is

$$u_1 + u_2 + u_3 + \ldots + u_n\,,$$

and we write $S_n$ for the sum of these $n$ terms. So although the ideas of a 'sequence' and a 'series' are related, there is an important distinction between them.

For example, let us consider the sequence of numbers

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad \ldots, \quad n\,.$$

Then $S_1 = 1$, as it is the sum of just the first term on its own. The sum of the first two terms is $S_2 = 1 + 2 = 3$. Continuing, we get

$$\begin{aligned} S_3 &= 1+2+3 &= 6\,, \\ S_4 &= 1+2+3+4 &= 10\,, \end{aligned}$$

and so on.

 

**Key Point**

A series is a sum of the terms in a sequence. If there are $n$ terms in the sequence and we evaluate the sum then we often write $S_n$ for the result, so that

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n \, .$$

**Exercise 2**

Write down $S_1$, $S_2, \ldots, S_n$ for the sequences

(a) $1, 3, 5, 7, 9, 11;$
(b) $4, 2, 0, -2, -4.$

# 3. Arithmetic progressions

Consider these two common sequences

$$1, \quad 3, \quad 5, \quad 7, \quad \ldots$$

and

$$0, \quad 10, \quad 20, \quad 30, \quad 40, \quad \ldots \, .$$

It is easy to see how these sequences are formed. They each start with a particular first term, and then to get successive terms we just add a fixed value to the previous term. In the first sequence we add 2 to get the next term, and in the second sequence we add 10. So the difference between consecutive terms in each sequence is a constant. We could also subtract a constant instead, because that is just the same as adding a negative constant. For example, in the sequence

$$8, \quad 5, \quad 2, \quad -1, \quad -4, \quad \ldots$$

the difference between consecutive terms is $-3$. Any sequence with this property is called an *arithmetic progression*, or AP for short.

We can use algebraic notation to represent an arithmetic progression. We shall let $a$ stand for the first term of the sequence, and let $d$ stand for the common difference between successive terms. For example, our first sequence could be written as

$$\begin{array}{llllll} 1, & 3, & 5, & 7, & 9, & \ldots \\ 1, & 1+2, & 1+2\times 2, & 1+3\times 2, & 1+4\times 2, & \ldots \quad , \end{array}$$

and this can be written as

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad a+4d, \quad \ldots$$

where $a = 1$ is the first term, and $d = 2$ is the common difference. If we wanted to write down the $n$-th term, we would have

$$a + (n-1)d \, ,$$

 

because if there are $n$ terms in the sequence there must be $(n-1)$ common differences between successive terms, so that we must add on $(n-1)d$ to the starting value $a$. We also sometimes write $\ell$ for the last term of a finite sequence, and so in this case we would have

$$\ell = a + (n-1)d\,.$$

**Key Point**

An arithmetic progression, or AP, is a sequence where each new term after the first is obtained by adding a constant $d$, called the *common difference*, to the preceding term. If the first term of the sequence is $a$ then the arithmetic progression is

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad \ldots$$

where the $n$-th term is $a+(n-1)d$.

**Exercise 3**

(a) Write down the first five terms of the AP with first term 8 and common difference 7.

(b) Write down the first five terms of the AP with first term 2 and common difference $-5$.

(c) What is the common difference of the AP $11, -1, -13, -25, \ldots$ ?

(d) Find the 17th term of the arithmetic progression with first term 5 and common difference 2.

(e) Write down the 10th and 19th terms of the APs
  (i) $8, 11, 14, \ldots$,
  (ii) $8, 5, 2 \ldots$.

(f) An AP is given by $k, 2k/3, k/3, 0, \ldots$.
  (i) Find the sixth term.
  (ii) Find the $n$ th term.
  (iii) If the 20th term is equal to 15, find $k$.

# 4. The sum of an arithmetic series

Sometimes we want to add the terms of a sequence. What would we get if we wanted to add the first $n$ terms of an arithmetic progression? We would get

$$S_n = a + (a+d) + (a+2d) + \ldots + (\ell - 2d) + (\ell - d) + \ell\,.$$

Now this is now a series, as we have added together the $n$ terms of a sequence. This is an *arithmetic series*, and we can find its sum by using a trick. Let us write the series down again, but this time we shall write it down with the terms in reverse order. We get

$$S_n = \ell + (\ell - d) + (\ell - 2d) + \ldots + (a+2d) + (a+d) + a\,.$$

 

We are now going to add these two series together. On the left-hand side, we just get $2S_n$. But on the right-hand side, we are going to add the terms in the two series so that each term in the first series will be added to the term vertically below it in the second series. We get

$$2S_n = (a+\ell) + (a+\ell) + (a+\ell) + \ldots + (a+\ell) + (a+\ell) + (a+\ell)\,,$$

and on the right-hand side there are $n$ copies of $(a+\ell)$ so we get

$$2S_n = n(a+\ell)\,.$$

But of course we want $S_n$ rather than $2S_n$, and so we divide by 2 to get

$$S_n = \tfrac{1}{2}n(a+\ell)\,.$$

We have found the sum of an arithmetic progression in terms of its first and last terms, $a$ and $\ell$, and the number of terms $n$.

We can also find an expression for the sum in terms of the $a$, $n$ and the common difference $d$. To do this, we just substitute our formula for $\ell$ into our formula for $S_n$. From

$$\ell = a + (n-1)d\,, \qquad S_n = \tfrac{1}{2}n(a+\ell)$$

we obtain

$$\begin{aligned} S_n &= \tfrac{1}{2}n(a + a + (n-1)d) \\ &= \tfrac{1}{2}n(2a + (n-1)d)\,. \end{aligned}$$

**Key Point**

The sum of the terms of an arithmetic progression gives an arithmetic series. If the starting value is $a$ and the common difference is $d$ then the sum of the first $n$ terms is

$$S_n = \tfrac{1}{2}n(2a + (n-1)d)\,.$$

If we know the value of the last term $\ell$ instead of the common difference $d$ then we can write the sum as

$$S_n = \tfrac{1}{2}n(a+\ell)\,.$$

**Example**

Find the sum of the first 50 terms of the sequence

$$1,\ 3,\ 5,\ 7,\ 9,\ \ldots\,.$$

**Solution**

This is an arithmetic progression, and we can write down

$$a = 1\,, \qquad d = 2\,, \qquad n = 50\,.$$

We now use the formula, so that

$$\begin{aligned} S_n &= \tfrac{1}{2}n(2a + (n-1)d) \\ S_{50} &= \tfrac{1}{2} \times 50 \times (2 \times 1 + (50-1) \times 2) \\ &= 25 \times (2 + 49 \times 2) \\ &= 25 \times (2 + 98) \\ &= 2500\,. \end{aligned}$$

**Example**

Find the sum of the series

$$1 + 3{\cdot}5 + 6 + 8{\cdot}5 + \ldots + 101\,.$$

**Solution**

This is an arithmetic series, because the difference between the terms is a constant value, $2{\cdot}5$. We also know that the first term is 1, and the last term is 101. But we do not know how many terms are in the series. So we will need to use the formula for the last term of an arithmetic progression,

$$\ell = a + (n-1)d$$

to give us

$$101 = 1 + (n-1) \times 2{\cdot}5\,.$$

Now this is just an equation for $n$, the number of terms in the series, and we can solve it. If we subtract 1 from each side we get

$$100 = (n-1) \times 2{\cdot}5$$

and then dividing both sides by $2{\cdot}5$ gives us

$$40 = n - 1$$

so that $n = 41$. Now we can use the formula for the sum of an arithmetic progression, in the version using $\ell$, to give us

$$\begin{aligned} S_n &= \tfrac{1}{2}n(a + \ell) \\ S_{41} &= \tfrac{1}{2} \times 41 \times (1 + 101) \\ &= \tfrac{1}{2} \times 41 \times 102 \\ &= 41 \times 51 \\ &= 2091\,. \end{aligned}$$

 

**Example**

An arithmetic progression has 3 as its first term. Also, the sum of the first 8 terms is twice the sum of the first 5 terms. Find the common difference.

**Solution**

We are given that $a = 3$. We are also given some information about the sums $S_8$ and $S_5$, and we want to find the common difference. So we shall use the formula

$$S_n = \tfrac{1}{2}n(2a + (n-1)d)$$

for the sum of the first $n$ terms. This tells us that

$$S_8 = \tfrac{1}{2} \times 8 \times (6 + 7d)\,.$$

and that

$$S_5 = \tfrac{1}{2} \times 5 \times (6 + 4d)$$

So, using the given fact that $S_8 = 2S_5$, we see that

$$\begin{aligned} \tfrac{1}{2} \times 8 \times (6+7d) &= 2 \times \tfrac{1}{2} \times 5 \times (6+4d) \\ 4 \times (6+7d) &= 5 \times (6+4d) \\ 24 + 28d &= 30 + 20d \\ 8d &= 6 \\ d &= \tfrac{3}{4}\,. \end{aligned}$$

**Exercise 4**

(a) Find the sum of the first 23 terms of the AP $4, -3, -10, \ldots$.

(b) An arithmetic series has first term 4 and common difference $\frac{1}{2}$. Find
   (i) the sum of the first 20 terms,
   (ii) the sum of the first 100 terms.

(c) Find the sum of the arithmetic series with first term 1, common difference 3, and last term 100.

(d) The sum of the first 20 terms of an arithmetic series is identical to the sum of the first 22 terms. If the common difference is $-2$, find the first term.

# 5. Geometric progressions

We shall now move on to the other type of sequence we want to explore.

Consider the sequence

$$2, \;\; 6, \;\; 18, \;\; 54, \;\; \ldots\,.$$

Here, each term in the sequence is 3 times the previous term. And in the sequence

$$1, \;\; -2, \;\; 4, \;\; -8, \;\; \ldots\,,$$

each term is $-2$ times the previous term. Sequences such as these are called *geometric progressions*, or GPs for short.

 

Let us write down a general geometric progression, using algebra. We shall take $a$ to be the first term, as we did with arithmetic progressions. But here, there is no common difference. Instead there is a common ratio, as the ratio of successive terms is always constant. So we shall let $r$ be this common ratio. With this notation, the general geometric progression can be expressed as

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad \ldots .$$

So the $n$-th can be calculated quite easily. It is $ar^{n-1}$, where the power $(n-1)$ is always one less than the position $n$ of the term in the sequence. In our first example, we had $a = 2$ and $r = 3$, so we could write the first sequence as

$$2, \quad 2 \times 3, \quad 2 \times 3^2, \quad 2 \times 3^3, \quad \ldots .$$

In our second example, $a = 1$ and $r = -2$, so that we could write it as

$$1, \quad 1 \times (-2), \quad 1 \times (-2)^2, \quad 1 \times (-2)^3, \ldots .$$

**Key Point**

A geometric progression, or GP, is a sequence where each new term after the first is obtained by multiplying the preceding term by a constant $r$, called the *common ratio*. If the first term of the sequence is $a$ then the geometric progression is

$$a, ar, ar^2, ar^3, \ldots$$

where the $n$-th term is $ar^{n-1}$.

**Exercise 5**

(a) Write down the first five terms of the geometric progression which has first term 1 and common ratio $\frac{1}{2}$.

(b) Find the 10th and 20th terms of the GP with first term 3 and common ratio 2.

(c) Find the 7th term of the GP $2, -6, 18, \ldots,$

# 6. The sum of a geometric series

Suppose that we want to find the sum of the first $n$ terms of a geometric progression. What we get is

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}\,,$$

and this is called a *geometric series*. Now the trick here to find the sum is to multiply by $r$ and then subtract:

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \\ rSn &= \phantom{a +{}} ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \\ S_n - rS_n &= a \phantom{{}+ ar + ar^2 + ar^3 + \ldots + ar^{n-1}} - ar^n \end{aligned}$$

so that

$$S_n(1-r) = a(1-r^n)\,.$$

Now divide by $1-r$ (as long as $r \neq 1$) to give

$$S_n = \frac{a(1-r^n)}{1-r}\,.$$

**Key Point**

The sum of the terms of a geometric progression gives a geometric series. If the starting value is $a$ and the common ratio is $r$ then the sum of the first $n$ terms is

$$S_n = \frac{a(1-r^n)}{1-r}$$

provided that $r \neq 1$.

**Example**

Find the sum of the geometric series

$$2 + 6 + 18 + 54 + \ldots$$

where there are 6 terms in the series.

**Solution**

For this series, we have $a = 2$, $r = 3$ and $n = 6$. So

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} \\ S_6 &= \frac{2(1-3^6)}{1-3} \\ &= \frac{2(1-729)}{-2} \\ &= -(-728) \\ &= 728\,. \end{aligned}$$

**Example**

Find the sum of the geometric series

$$8 - 4 + 2 - 1 + \ldots$$

where there are 5 terms in the series.

 

**Solution**

For this series, we have $a = 8$, $r = -\frac{1}{2}$ and $n = 5$. So

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} \\ S_5 &= \frac{8(1-\left(-\frac{1}{2}\right)^5)}{1-\left(-\frac{1}{2}\right)} \\ &= \frac{8(1-\left(-\frac{1}{32}\right))}{\frac{3}{2}} \\ &= \frac{2 \times 8 \times \frac{33}{32}}{3} \\ &= \tfrac{11}{2} \\ &= 5\tfrac{1}{2}\,. \end{aligned}$$

**Example**

How many terms are there in the geometric progression

$$2, \quad 4, \quad 8, \quad \ldots, \quad 128\,?$$

**Solution**

In this sequence $a = 2$ and $r = 2$. We also know that the $n$-th term is 128. But the formula for the $n$-th term is $ar^{n-1}$. So

$$\begin{aligned} 128 &= 2 \times 2^{n-1} \\ 64 &= 2^{n-1} \\ 2^6 &= 2^{n-1} \\ 6 &= n-1 \\ n &= 7\,. \end{aligned}$$

So there are 7 terms in this geometric progression.

**Example**

How many terms in the geometric progression

$$1, \quad 1{\cdot}1, \quad 1{\cdot}21, \quad 1{\cdot}331, \quad \ldots$$

will be needed so that the sum of the first $n$ terms is greater than 20?

**Solution**

The sequence is a geometric progression with $a = 1$ and $r = 1{\cdot}1$. We want to find the smallest value of $n$ such that $S_n > 20$. Now

$$S_n = \frac{a(1-r^n)}{1-r}\,,$$

 

so

$$\begin{aligned}
\frac{1 \times (1 - 1{\cdot}1^n)}{1 - 1{\cdot}1} &> 20 \\
\frac{1 - 1{\cdot}1^n}{-0{\cdot}1} &> 20 \\
(1{\cdot}1^n - 1) \times 10 &> 20 \\
1{\cdot}1^n - 1 &> 2 \\
1{\cdot}1^n &> 3\,.
\end{aligned}$$

If we now take logarithms of both sides, we get

$$n \ln 1{\cdot}1 > \ln 3$$

and as $\ln 1{\cdot}1 > 0$ we obtain

$$n > \ln 3 / \ln 1{\cdot}1 = 11.5267\ldots$$

and therefore the smallest whole number value of $n$ is 12.

**Exercise 6**

(a) Find the sum of the first five terms of the GP with first term 3 and common ratio 2.

(b) Find the sum of the first 20 terms of the GP with first term 3 and common ratio 1.5.

(c) The sum of the first 3 terms of a geometric series is $\frac{37}{8}$. The sum of the first six terms is $\frac{3367}{512}$. Find the first term and common ratio.

(d) How many terms in the GP $4, 3.6, 3.24, \ldots$ are needed so that the sum exceeds 35?

# 7. Convergence of geometric series

Consider the geometric progression

$$1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \tfrac{1}{16},\ \ldots\,.$$

We have $a = 1$ and $r = \frac{1}{2}$, and so we can calculate some sums. We get

$$\begin{aligned}
S_1 &= & &1 \\
S_2 &= 1 + \tfrac{1}{2} &= &\tfrac{3}{2} \\
S_3 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} &= &\tfrac{7}{4} \\
S_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} &= &\tfrac{15}{8} \\
&\vdots
\end{aligned}$$

and there seems to be a pattern because

$$\begin{aligned}
1 &= 2 - 1 \\
\tfrac{3}{2} &= 2 - \tfrac{1}{2} \\
\tfrac{7}{4} &= 2 - \tfrac{1}{4} \\
\tfrac{15}{8} &= 2 - \tfrac{1}{8}\,.
\end{aligned}$$

In each case, we subtract a small quantity from 2, and as we take successive sums the quantity gets smaller and smaller. If we were able to add 'infinitely many' terms, then the answer 'ought to be' 2 — or as near as we want to get to 2.

Let us see if we can explain this by using some algebra. We know that

$$S_n = \frac{a(1-r^n)}{1-r},$$

and we want to examine this formula in the case of our particular example where $r = \frac{1}{2}$. Now the formula contains the term $r^n$ and, as $-1 < r < 1$, this term will get closer and closer to zero as $n$ gets larger and larger. So, if $-1 < r < 1$, we can say that the 'sum to infinity' of a geometric series is

$$S_\infty = \frac{a}{1-r},$$

where we have omitted the term $r^n$. We say that this is the *limit* of the sums $S_n$ as $n$ 'tends to infinity'. You will find more details of this concept in another unit.

**Example** Find the sum to infinity of the geometric progression

$$1,\ \tfrac{1}{3},\ \tfrac{1}{9},\ \tfrac{1}{27},\ \ldots\,.$$

**Solution** For this geometric progression we have $a = 1$ and $r = \frac{1}{3}$. As $-1 < r < 1$ we can use the formula, so that

$$S_\infty = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \tfrac{3}{2}.$$

**Key Point**

The sum to infinity of a geometric progression with starting value $a$ and common ratio $r$ is given by

$$S_\infty = \frac{a}{1-r}$$

where $-1 < r < 1$.

**Exercise 7**

(a) Find the sum to infinity of the GP with first term 3 and common ratio $\frac{1}{2}$.

(b) The sum to infinity of a GP is four times the first term. Find the common ratio.

(c) The sum to infinity of a GP is twice the sum of the first two terms. Find possible values of the common ratio.

 

**Answers**

1.

(a) 8, 11, 14, 17, 20

(b) 1, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$; tenth term is $\frac{1}{100}$

(c) 1, 1, 2, 3, 5, 8, 13, 21

(d) 1, $-\frac{1}{2}$, $\frac{1}{3}$, $-\frac{1}{4}$, $\frac{1}{5}$

2.

(a) 1, 4, 9, 16, 25, 36

(b) $4,\quad 6,\quad 6,\quad 4,\quad 0$

3.

(a) 8, 15, 22, 29, 36

(b) $2,\quad -3,\quad -8,\quad -13,\quad -18$

(c) $-12$

(d) 37

(e) (i) 35, 62 (ii) $-19$, $-46$

(f) (i) $-2k/3$ (ii) $k(4-n)/3$ (iii) $-\frac{45}{16}$

4.

(a) $-1679$ (b) (i) 175, (ii) 2875 (c) 1717 (d) 41

5.

(a) 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$ (b) 1536, 1,572,864 (c) 1458

6.

(a) 93 (b) 19,946 (c) 2, $\frac{3}{4}$ (d) 20 terms

7.

(a) 6 (b) $\frac{3}{4}$ (c) $\pm 1/\sqrt{2}$