Trigonometric equations

mc-TY-trigeqn-2009-1

In this unit we consider the solution of trigonometric equations. The strategy we adopt is to find one solution using knowledge of commonly occuring angles, and then use the symmetries in the graphs of the trigonometric functions to deduce additional solutions. Familiarity with the graphs of these functions is essential.

In order to master the techniques explained here it is vital that you undertake the practice exercises provided.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- find solutions of trigonometric equations
- use trigonometric identities in the solution of trigonometric equations

Contents

1. Introduction

This unit looks at the solution of trigonometric equations. In order to solve these equations we shall make extensive use of the graphs of the functions sine, cosine and tangent. The symmetries which are apparent in these graphs, and their periodicities are particularly important as we shall see.

2. Some special angles and their trigonometric ratios.

In the examples which follow a number of angles and their trigonometric ratios are used frequently. We list these angles and their sines, cosines and tangents.

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

3. Some simple trigonometric equations

Example

Suppose we wish to solve the equation $\sin x = 0.5$ and we look for all solutions lying in the interval $0° \leq x \leq 360°$. This means we are looking for all the angles, x, in this interval which have a sine of 0.5.

We begin by sketching a graph of the function $\sin x$ over the given interval. This is shown in Figure 1.

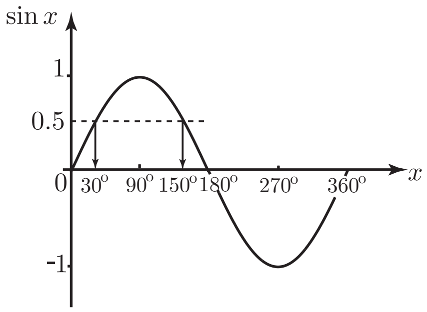

Figure 1. A graph of $\sin x$.

We have drawn a dotted horizontal line on the graph indicating where $\sin x = 0.5$. The solutions of the given equation correspond to the points where this line crosses the curve. From the Table above we note that the first angle with a sine equal to 0.5 is $30°$. This is indicated in Figure 1. Using the symmetries of the graph, we can deduce all the angles which have a sine of 0.5. These are:

$$x = 30°, 150°$$

This is because the second solution, $150°$, is the same distance to the left of $180°$ that the first is to the right of $0°$. There are no more solutions within the given interval.

Example

Suppose we wish to solve the equation $\cos x = -0.5$ and we look for all solutions lying in the interval $0 \leq x \leq 360°$.

As before we start by looking at the graph of $\cos x$. This is shown in Figure 2. We have drawn a dotted horizontal line where $\cos x = -0.5$. The solutions of the equation correspond to the points where this line intersects the curve. One fact we do know from the Table on page 2 is that $\cos 60° = +0.5$. This is indicated on the graph. We can then make use of the symmetry to deduce that the first angle with a cosine equal to -0.5 is $120°$. This is because the angle must be the same distance to the right of $90°$ that $60°$ is to the left. From the graph we see, from consideration of the symmetry, that the remaining solution we seek is $240°$. Thus

$$x = 120°, 240°$$

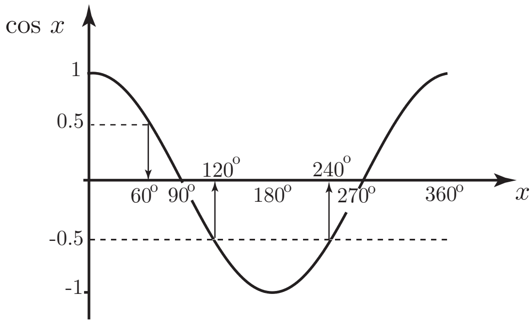

Figure 2. A graph of $\cos x$.

Example

Suppose we wish to solve $\sin 2x = \dfrac{\sqrt{3}}{2}$ for $0 \leq x \leq 360°$.

Note that in this case we have the sine of a multiple angle, $2x$.

To enable us to cope with the multiple angle we shall consider a new variable u where $u = 2x$, so the problem becomes that of solving

$$\sin u = \frac{\sqrt{3}}{2} \qquad \text{for } 0 \leq u \leq 720°$$

We draw a graph of $\sin u$ over this interval as shown in Figure 3.

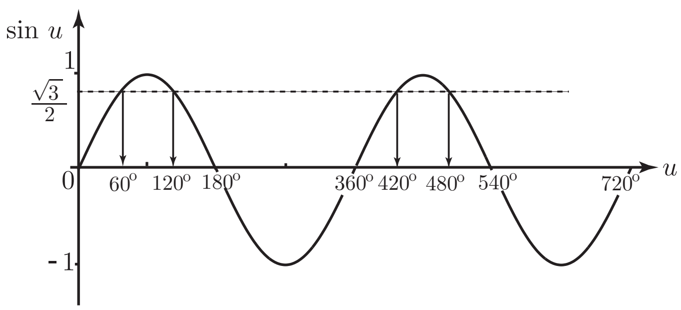

Figure 3. A graph of $\sin u$ for u lying between 0 and 720°.

By referring to the Table on page 2 we know that $\sin 60° = \frac{\sqrt{3}}{2}$. This is indicated on the graph. From the graph we can deduce another angle which has a sine of $\frac{\sqrt{3}}{2}$. This is $120°$. Because of the periodicity we can see there are two more angles, $420°$ and $480°$. We therefore know all the angles in the interval with sine equal to $\frac{\sqrt{3}}{2}$, namely

$$u = 60°, 120°, 420°, 480°$$

But $u = 2x$ so that

$$2x = 60°, 120°, 420°, 480°$$

from which

$$x = 30°, 60°, 210°, 240°$$

Example

Suppose we wish to solve $\tan 3x = -1$ for values of x in the interval $0° \leq x \leq 180°$.

Note that in this example we have the tangent of a multiple angle, $3x$.

To enable us to cope with the multiple angle we shall consider a new variable u where $u = 3x$, so the problem becomes that of solving

$$\tan u = -1 \qquad \text{for } 0 \leq u \leq 540°$$

We draw a graph of $\tan u$ over this interval as shown in Figure 4.

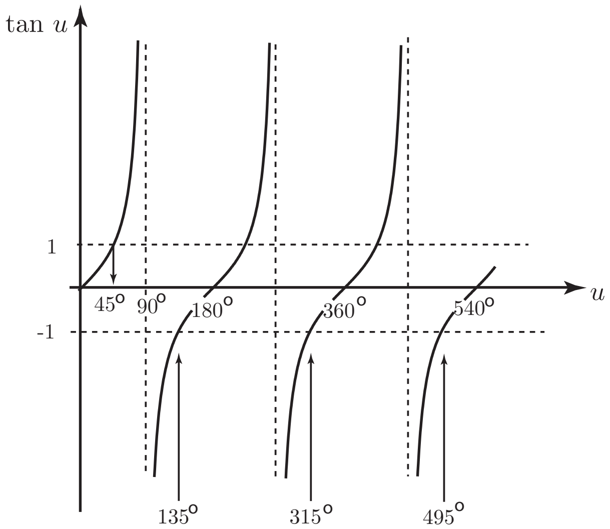

Figure 4. A graph of $\tan u$.

We know from the Table on page 2 that an angle whose tangent is 1 is $45°$, so using the symmetry in the graph we can find the angles which have a tangent equal to -1. The first will be the same distance to the right of $90°$ that $45°$ is to the left, that is $135°$. The other angles will each be $180°$ further to the right because of the periodicity of the tangent function. Consequently the solutions of $\tan u = -1$ are given by

$$u = 135°, 315°, 495°,$$

But $u = 3x$ and so

$$3x = 135°, 315°, 495°,$$

from which

$$x = 45°, 105°, 165°$$

Example

Suppose we wish to solve $\cos\dfrac{x}{2} = -\dfrac{1}{2}$ for values of x in the interval $0 \le x \le 360°$.

In this Example we are dealing with the cosine of a multiple angle, $\dfrac{x}{2}$.

To enable us to handle this we make a substitution $u = \dfrac{x}{2}$ so that the equation becomes

$$\cos u = -\frac{1}{2} \qquad \text{for } 0 \le u \le 180°$$

A graph of $\cos u$ over this interval is shown in Figure 5.

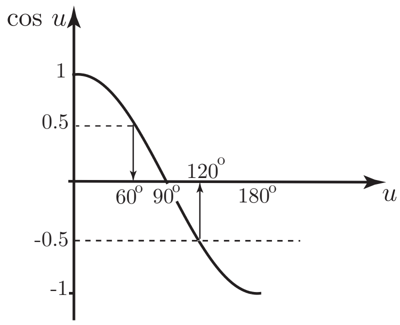

Figure 5. A graph of $\cos u$.

We know that the angle whose cosine is $\frac{1}{2}$ is $60°$. Using the symmetry in the graph we can find all the angles with a cosine equal to $-\frac{1}{2}$. In the interval given there is only one angle with cosine equal to $-\frac{1}{2}$ and that is $u = 120°$

But $u = \dfrac{x}{2}$ and so $x = 2u$. We conclude that there is a single solution, $x = 240°$.

Let us now look at some examples over the interval $-180° \le x \le 180°$.

Example

Suppose we wish to solve $\sin x = 1$ for $-180° \le x \le 180°$.

From the graph of $\sin x$ over this interval, shown in Figure 6, we see there is only one angle which has a sine equal to 1, that is $x = 90°$.

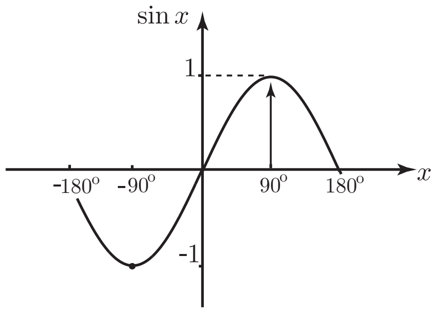

Figure 6. A graph of the sine function

Example

Suppose we wish to solve $\cos 2x = \dfrac{1}{2}$ for $-180° \leq x \leq 180°$.

In this Example we have a multiple angle, $2x$.

To handle this we let $u = 2x$ and instead solve

$$\cos u = \frac{1}{2} \qquad \text{for } -360° \leq x \leq 360°$$

A graph of the cosine function over this interval is shown in Figure 7.

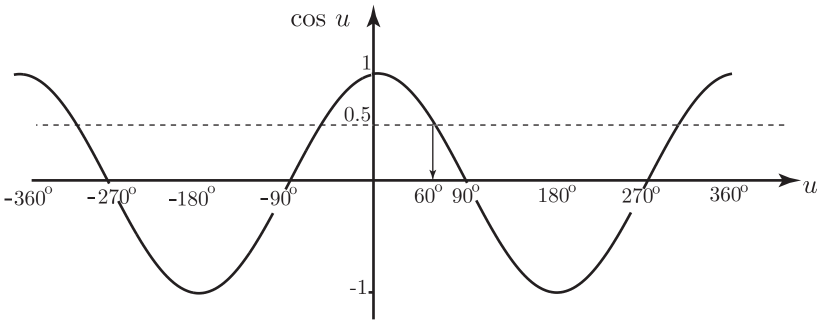

Figure 7. A graph of $\cos u$.

The dotted line indicates where the cosine is equal to $\frac{1}{2}$. Remember we already know one angle which has cosine equal to $\frac{1}{2}$ and this is $60°$. From the graph, and making use of symmetry, we can deduce all the other angles with cosine equal to $\frac{1}{2}$. These are

$$u = -300°, -60°, 60°, 300°$$

Then $u = 2x$ so that

$$2x = -300°, -60°, 60°, 300°$$

from which

$$x = -150°, -30°, 30°, 150°$$

Example

Suppose we wish to solve $\tan 2x = \sqrt{3}$ for $-180° \leq x \leq 180°$.

We again have a multiple angle, $2x$. We handle this by letting $u = 2x$ so that the problem becomes that of solving

$$\tan u = \sqrt{3} \qquad \text{for } -360° \leq u \leq 360°$$

We plot a graph of $\tan u$ between $-360°$ and $360°$ as shown in Figure 8.

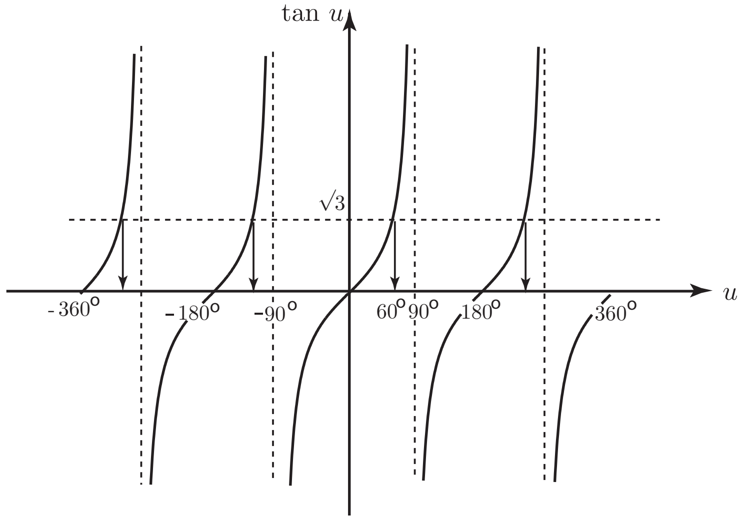

Figure 8. A graph of $\tan u$.

We know from the Table on page 2 that one angle which has a tangent equal to $\sqrt{3}$ is $u = 60°$. We can use the symmetry of the graph to deduce others. These are

$$u = -300°, -120°, 60°, 240°$$

But $u = 2x$ and so

$$2x = -300°, -120°, 60°, 240°$$

and so the required solutions are

$$x = -150°, -60°, 30°, 120°$$

Exercise 1

1. Find all the solutions of each of the following equations in the given range

   (a) $\sin x = \dfrac{1}{\sqrt{2}}$ for $0 < x < 360^{\text{o}}$

   (b) $\cos x = -\dfrac{1}{\sqrt{2}}$ for $0 < x < 360^{\text{o}}$

   (c) $\tan x = \dfrac{1}{\sqrt{3}}$ for $0 < x < 360^{\text{o}}$

   (d) $\cos x = -1$ for $0 < x < 360^{\text{o}}$

2. Find all the solutions of each of the following equations in the given range

   (a) $\tan x = \sqrt{3}$ for $-180^{\text{o}} < x < 180^{\text{o}}$

   (b) $\tan x = -\sqrt{3}$ for $-180^{\text{o}} < x < 180^{\text{o}}$

   (c) $\cos x = \dfrac{1}{2}$ for -180$^{\text{o}} < x < 180^{\text{o}}$

   (d) $\sin x = -\dfrac{1}{\sqrt{2}}$ for $-180^{\text{o}} < x < 180^{\text{o}}$

3. Find all the solutions of each of the following equations in the given range

   (a) $\cos 2x = \dfrac{1}{\sqrt{2}}$ for $-180^{\rm o} < x < 180^{\rm o}$

   (b) $\tan 3x = 1$ for $-90^{\rm o} < x < 90^{\rm o}$

   (c) $\sin 2x = \dfrac{1}{2}$ for $-180^{\rm o} < x < 0$

   (d) $\cos \frac{1}{2}x = -\dfrac{\sqrt{3}}{2}$ for $-180^{\rm o} < x < 180^{\rm o}$

4. Using identities in the solution of equations

There are many **trigonometric identities**. Two commonly occuring ones are

$$\sin^2 x + \cos^2 x = 1 \qquad\qquad \sec^2 x = 1 + \tan^2 x$$

We will now use these in the solution of trigonometric equations. (If necessary you should refer to the unit entitled *Trigonometric Identities*).

Example

Suppose we wish to solve the equation $\cos^2 x + \cos x = \sin^2 x$ for $0^\circ \leq x \leq 180^\circ$.

We can use the identity $\sin^2 x + \cos^2 x = 1$, rewriting it as $\sin^2 x = 1 - \cos^2 x$ to write the given equation entirely in terms of cosines.

$$\begin{aligned} \cos^2 x + \cos x &= \sin^2 x \\ \cos^2 x + \cos x &= 1 - \cos^2 x \end{aligned}$$

Rearranging, we can write

$$2\cos^2 x + \cos x - 1 = 0$$

This is a quadratic equation in which the variable is $\cos x$. This can be factorised to

$$(2\cos x - 1)(\cos x + 1) = 0$$

Hence

$$2\cos x - 1 = 0 \qquad \text{or} \qquad \cos x + 1 = 0$$

from which

$$\cos x = \frac{1}{2} \qquad \text{or} \qquad \cos x = -1$$

We solve each of these equations in turn. By referring to the graph of $\cos x$ over the interval $0 \leq x \leq 180^\circ$ which is shown in Figure 9, we see that there is only one solution of the equation $\cos x = \frac{1}{2}$ in this interval, and this is $x = 60^\circ$. From the same graph we can deduce the solution of $\cos x = -1$ to be $x = 180^\circ$.

So there are two solutions of the original equation, $60°$ and $180°$.

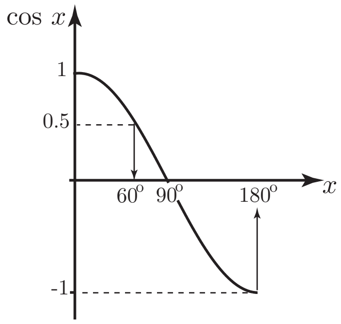

Figure 9. A graph of $\cos x$.

Example

Suppose we wish to solve the equation $3\tan^2 x = 2\sec^2 x + 1$ for $0° \leq x \leq 180°$.

In this example we will simplify the equation using the identity $\sec^2 x = 1 + \tan^2 x$.

$$\begin{aligned} 3\tan^2 x &= 2\sec^2 x + 1 \\ 3\tan^2 x &= 2(1 + \tan^2 x) + 1 \\ 3\tan^2 x &= 2 + 2\tan^2 x + 1 \end{aligned}$$

Rearranging we can write

$$\tan^2 x = 3$$

so that

$$\tan x = +\sqrt{3} \text{ or } -\sqrt{3}$$

We solve each of these equations separately.

The solutions of $\tan x = \sqrt{3}$ can be obtained by inspecting the graph in Figure 10. From the Table on page 2 we know that one angle with a tangent of $\sqrt{3}$ is $60°$. There are no other solutions in the given interval. Using the symmetry of the graph we can deduce the solution of the equation $\tan x = -\sqrt{3}$. This is $x = 120°$.

So the given equation has two solutions, $x = 60°$ and $x = 120°$.

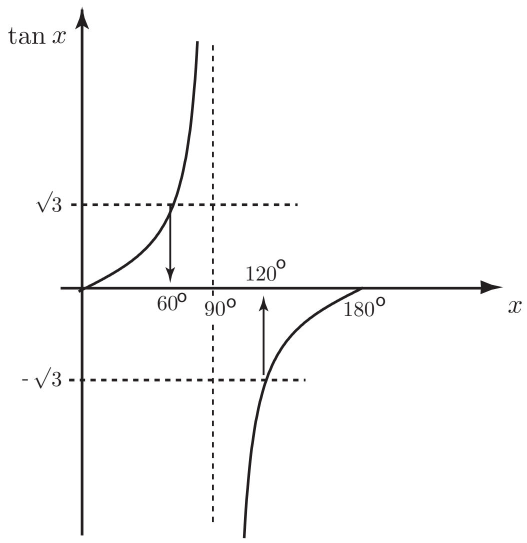

Figure 10. A graph of $\tan x$.

Exercise 2

1. Find all the solutions of each of the following equations in the given range

   (a) $3\cos^2 x - 3 = \sin^2 x - 1$ for $0 < x < 360^{\text{o}}$

   (b) $5\cos^2 x = 4 - 3\sin^2 x$ for -180$^{\text{o}}$$< x < 180^{\text{o}}$

   (c) $\tan^2 x = 2\sec^2 x - 3$ for -180$^{\text{o}}$$< x < 180^{\text{o}}$

   (d) $\cos^2 x + 3\cos x = \sin^2 x - 2$ for $0 < x < 360^{\text{o}}$

5. Some examples where the interval is given in radians

In the previous examples, we sought solutions of equations where the angle required was measured in degrees. We now look at some examples where the angle is measured in radians. In fact, it is advisable to work with angles in radians because many trigonometric equations only make sense when an angle is measured in this way.

Example

Suppose we wish to solve the equation $\tan x = -1$ for $-\pi \leq x \leq \pi$.

A graph of the tangent function over this interval is shown in Figure 11. We know from the Table on page 2 that one angle with a tangent equal to 1 is $\frac{\pi}{4}$. Using the symmetry of the graph we can deduce that the solutions of the equation $\tan x = -1$ are

$$x = -\frac{\pi}{4},\ \frac{3\pi}{4}$$

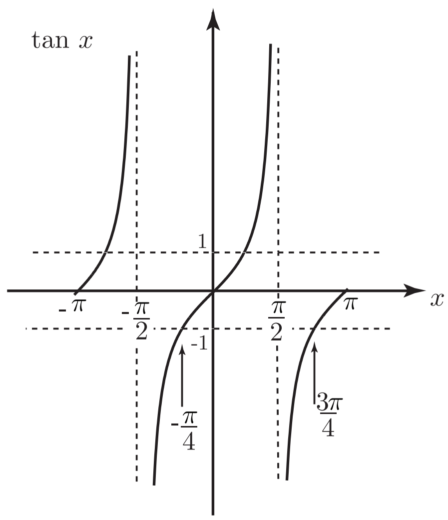

Figure 11. A graph of $\tan x$.

Example

Suppose we wish to solve the equation $\cos 2x = \dfrac{\sqrt{3}}{2}$ for $0 \leq x \leq 2\pi$.

We handle the multiple angle by letting $u = 2x$ so that the problem becomes that of solving

$$\cos u = \frac{\sqrt{3}}{2} \qquad \text{for } 0 \leq u \leq 4\pi$$

So we have plotted a graph of $\cos u$ over this interval in Figure 12.

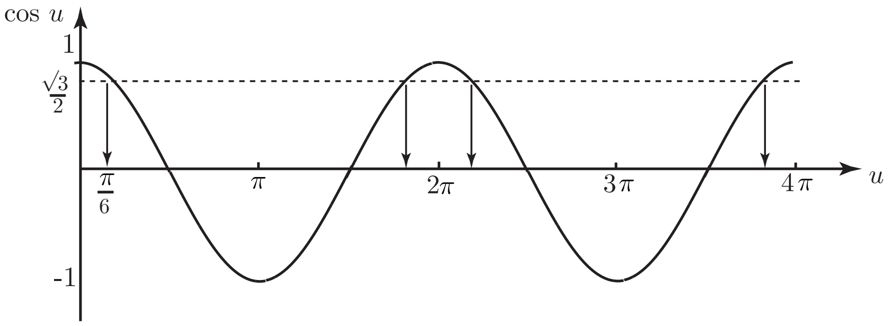

Figure 12. A graph of $\cos u$.

We know from the Table on page 2 that an angle which has cosine equal to $\frac{\sqrt{3}}{2}$ is $30°$, that is $u = \frac{\pi}{6}$. This is indicated on the graph. Using the symmetry of the graph we can deduce all the angles which have cosine equal to $\frac{\sqrt{3}}{2}$. These are

$$u = \frac{\pi}{6},\ 2\pi - \frac{\pi}{6},\ 2\pi + \frac{\pi}{6},\ 4\pi - \frac{\pi}{6}$$

But $u = 2x$ and so

$$\begin{aligned} 2x &= \frac{\pi}{6},\ 2\pi - \frac{\pi}{6},\ 2\pi + \frac{\pi}{6},\ 4\pi - \frac{\pi}{6} \\ &= \frac{\pi}{6},\ \frac{11\pi}{6},\ \frac{13\pi}{6},\ \frac{23\pi}{6} \end{aligned}$$

and so

$$x = \frac{\pi}{12},\ \frac{11\pi}{12},\ \frac{13\pi}{12},\ \frac{23\pi}{12}$$

Example

Suppose we wish to solve $\sin\frac{x}{2} = -\frac{\sqrt{3}}{2}$ for values of x in the interval $-\pi \leq x \leq \pi$.

We handle the multiple angle by letting $\frac{x}{2} = u$ so that the problem becomes that of solving

$$\sin u = -\frac{\sqrt{3}}{2} \qquad \text{for } -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$$

A graph of $\sin u$ over the given interval is shown in Figure 13.

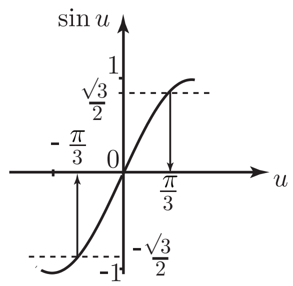

Figure 13. A graph of $\sin u$

We know from the Table on page 2 that an angle which has sine equal to $\frac{\sqrt{3}}{2}$ is $u = 60°$ or $u = \frac{\pi}{3}$. Using the symmetry of the graph we can deduce angles which have a sine equal to $-\frac{\sqrt{3}}{2}$. There is only one solution in the given interval and this is $u = -\frac{\pi}{3}$. But $u = \frac{x}{2}$ and so

$$\frac{x}{2} = -\frac{\pi}{3}$$

Hence

$$x = -\frac{2\pi}{3}$$

Example

Suppose we wish to solve the equation $2\cos^2 x + \sin x = 1$ for $0 \leq x \leq 2\pi$.

We shall use the identity $\sin^2 x + \cos^2 x = 1$ to rewrite the equation entirely in terms of sines.

$$\begin{aligned} 2\cos^2 x + \sin x &= 1 \\ 2(1 - \sin^2 x) + \sin x &= 1 \\ 2 - 2\sin^2 x + \sin x &= 1 \end{aligned}$$

and rearranging

$$2\sin^2 x - \sin x - 1 = 0$$

This is a quadratic equation in $\sin x$ which can be factorised to give

$$(2\sin x + 1)(\sin x - 1) = 0$$

Hence

$$2\sin x + 1 = 0 \qquad \text{and} \qquad \sin x - 1 = 0$$

from which

$$\sin x = -\frac{1}{2} \qquad \text{and} \qquad \sin x = 1$$

We solve each of these separately. Graphs of $\sin x$ are shown in Figure 14. From Figure 14(a) we can deduce the solution of $\sin x = 1$ to be $x = \frac{\pi}{2}$. Solutions of $\sin x = -\frac{1}{2}$ can be deduced from Figure 14(b). We know from the Table on page 2 that an angle with sine equal to $\frac{1}{2}$ is $30°$ or $\frac{\pi}{6}$. Using the symmetry in the graph we can deduce the angles with sine equal to $-\frac{1}{2}$ to be $\pi + \frac{\pi}{6}$ and $2\pi - \frac{\pi}{6}$. Hence

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

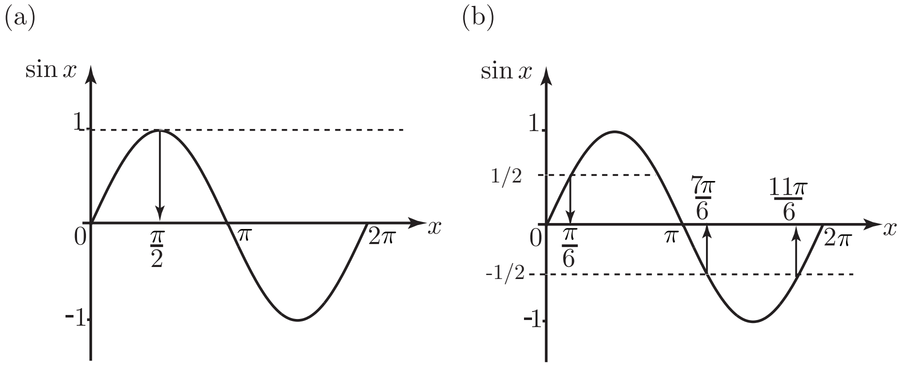

Figure 14. Graphs of the function $\sin x$

So, the full set of solutions of the given equation is $x = \dfrac{\pi}{2},\ \dfrac{7\pi}{6},\ \dfrac{11\pi}{6}$.

So, we have seen a large number of examples of the solution of trigonometric equations. The strategy is to obtain an initial solution and then work with the graph and its symmetries to find additional solutions.

Exercise 3

1. Find all the solutions of each of the following equations in the given range

   (a) $\tan x = \sqrt{3}$ for $0 < x < 2\pi$

   (b) $\sin 2x = -1$ for $-\pi < x < \pi$

   (c) $\cos 3x = \dfrac{1}{\sqrt{2}}$ for $-\pi < x < \pi$

   (d) $\tan \dfrac{1}{2}x = -1$ for $-2\pi < x < 0$

2. Find all the solutions of each of the following equations in the given range

   (a) $3\tan^2 x - 2 = 5\sec^2 x - 9$ for $0 < x < 2\pi$

   (b) $3\cos^2 x - 6\cos x = \sin^2 x - 3$ for $-\pi < x < \pi$

Answers

Exercise 1

1. a) 45^o, 135^o b) 135^o, 225^o c) 30^o, 210^o d) 180^o
2. a) -120^o, 60^o b) -60^o, 120^o c) -60^o, 60^o d) -135^o, -45^o
3. a) -157.5^o, -22.5^o, 22.5^o, 157.5^o b) -45^o, 15^o, 75^o c) -165^o, -105^o d) No solution

Exercise 2

1. a) 30^o, 150^o, 210^o, 330^o b) -135^o, -45^o, 45^o, 135^o
c) -135^o, -45^o, 45^o, 135^od) 120^o, 180^o, 240^o

Exercise 3

1. a) $\dfrac{\pi}{3}, \dfrac{4\pi}{3}$ b) $-\dfrac{\pi}{4}, \dfrac{3\pi}{4}$ c) $-\dfrac{3\pi}{4}, -\dfrac{7\pi}{12}, -\dfrac{\pi}{12}, \dfrac{\pi}{12}, \dfrac{7\pi}{12}, \dfrac{3\pi}{4}$ d) $-\dfrac{\pi}{2}$
2. a) $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$ b) $-\dfrac{\pi}{3}, 0, \dfrac{\pi}{3}$