

## The determinant of a $3 \times 3$ matrix

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We have seen that determinants are important in the solution of simultaneous equations and in finding inverses of matrices. The rule for evaluating the determinant of  $2 \times 2$  matrices is quite straightforward (if rather unexpected). To evaluate the determinant of a  $3 \times 3$  matrix is somewhat more complicated and relies on some other quantities known as minors and cofactors.

Throughout this leaflet we will work with the  $3 \times 3$  matrix

$$A = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix}$$

### Minors

Each element in a square matrix has its own **minor**. The minor is the value of the determinant of the matrix that results from crossing out the row and column of the element under consideration.

For the element 7 in matrix  $A$ , since this element is in the first row and first column, we cross out the first row and column of  $A$  to leave the  $2 \times 2$  matrix  $\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$ . We now evaluate the determinant of this matrix:

$$\begin{vmatrix} 3 & -1 \\ 4 & -2 \end{vmatrix} = (3 \times -2) - (-1 \times 4) = -6 - (-4) = -2$$

The minor of the element 7 is thus  $-2$ .

For the element 4, since this element is in the third row and second column, we cross out the third row and second column of  $A$  to leave the  $2 \times 2$  matrix  $\begin{pmatrix} 7 & 1 \\ 0 & -1 \end{pmatrix}$ . We now evaluate the determinant of this matrix:

$$\begin{vmatrix} 7 & 1 \\ 0 & -1 \end{vmatrix} = (7 \times -1) - (1 \times 0) = -7 - (0) = -7$$

The minor of the element 4 is thus  $-7$ .

We could continue in this way to find the minor of every element.

### Place signs

Each element in a square matrix has a **place sign**. The place sign of the element in the first row and first column is always '+'. The place signs then alternate from '+' to '-' as we move across the rows and down the columns. So, for a  $3 \times 3$  matrix the place signs are:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

### Cofactors

Each element in a square matrix has its own **cofactor**. The cofactor is the product of the element's place sign and minor.

The element 7 in matrix  $A$  has place sign  $+$  and minor  $-2$  so its cofactor is  $+(-2) = -2$ .

The element 4 in matrix  $A$  has place sign  $-$  and minor  $-7$  so its cofactor is  $-(-7) = 7$ .

Proceeding in this way we can find all the cofactors.

The original matrix, its matrix of minors and its matrix of cofactors are:

$$A = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix} \quad M = \begin{pmatrix} -2 & -3 & 9 \\ -8 & -11 & 34 \\ -5 & -7 & 21 \end{pmatrix} \quad C = \begin{pmatrix} -2 & 3 & 9 \\ 8 & -11 & -34 \\ -5 & 7 & 21 \end{pmatrix}$$

### Determinant of a $3 \times 3$ matrix

To evaluate the determinant of a  $3 \times 3$  matrix we choose any row or column of the matrix - this will contain three elements. We then find three products by multiplying each element in the row or column we have chosen by its cofactor. Finally, we sum these three products to find the value of the determinant. Note that it does not matter which row or column we choose, we will always get the same value for the determinant.

For our matrix  $A$ , suppose we choose the first row.

The three elements are 7, 2 and 1. The corresponding cofactors are  $-2$ , 3 and 9. The three products are therefore  $7 \times -2 = -14$ ,  $2 \times 3 = 6$ ,  $1 \times 9 = 9$ . We add these three products to get  $-14 + 6 + 9 = 1$ . The determinant of  $A$  is thus 1.

You might suspect that if we had chosen a different row or column we might have got a different answer. Let's try again. Suppose we choose the second column, 2, 3, 4, with corresponding cofactors 3,  $-11$ , 7. The three products are then  $2 \times 3 = 6$ ,  $3 \times -11 = -33$  and  $4 \times 7 = 28$ . We add these products to get  $6 - 33 + 28 = 1$ .

Whichever row or column we start with, the determinant found for this matrix will always be 1. Pick another row or column for yourself and check that you get the answer 1.

### A handy tip

When you need to evaluate the determinant of a matrix you do not need to find all the nine cofactors. You only need to find the cofactors of the three elements in the row or column that you have chosen. If any of the elements in the row or column that you have chosen are zero then you do not need to find their cofactors since when you work out the determinant the cofactor will be multiplied by the element and any value multiplied by zero is zero.

In the next leaflet in the series we will provide an example of finding a determinant in the most efficient way.

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Note that a video tutorial covering the content of this leaflet is available from **sigma**.