

# The gradient of a straight line segment

In this unit we find the gradient of a straight line segment, and the relationships between the gradients of parallel lines and of perpendicular lines.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- calculate the gradient of a line through two points;
- use gradients to determine whether two lines are parallel;
- use gradients to determine whether two lines are perpendicular.

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# 1. The gradient of a line

The gradient of a line segment is a measure of how steep the line is. A line with a large gradient will be steep; a line with a small gradient will be relatively shallow; and a line with zero gradient will be horizontal.

We can quantify this precisely as follows.

Consider Figure 1 which shows three line segments. The line segment  $AD$  is steeper than the line segment  $AC$ . In turn, this is steeper than  $AB$  which is horizontal. We can quantify this steepness mathematically by measuring the relative changes in  $x$  and  $y$  as we move from the beginning to the end of the line segment.

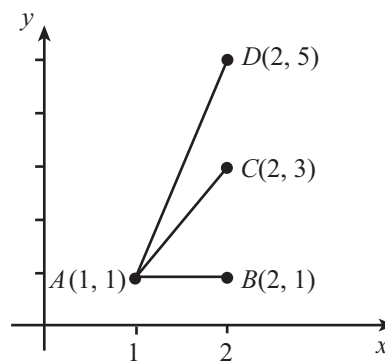


Figure 1.

On the segment  $AD$ ,  $y$  changes from 1 to 5 as  $x$  changes from 1 to 2. So the change in  $y$  is 4, and the change in  $x$  is 1. The relative change is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{5 - 1}{2 - 1} = \frac{4}{1} = 4$$

On the segment  $AC$ ,  $y$  changes from 1 to 3 as  $x$  changes from 1 to 2. So the change in  $y$  is 2, and the change in  $x$  is 1. The relative change is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{3 - 1}{2 - 1} = \frac{2}{1} = 2$$

On the segment  $AB$ ,  $y$  does not change as  $x$  changes from 1 to 2. So the change in  $y$  is 0, and the change in  $x$  is 1. The relative change is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{1 - 1}{2 - 1} = \frac{0}{1} = 0$$

The relative change,  $\frac{\text{change in } y}{\text{change in } x}$  is the **gradient** of the line segment. By defining the gradient in this way we see that this is consistent with our ideas of steepness - lines which are steeper have a larger gradient than lines which are less steep. Lines which are horizontal have zero gradient.

In the general case, if we take two points  $A(x_1, y_1)$  and  $B(x_2, y_2)$  (Figure 2) then we can work out the gradient using the same method as before, by finding the point  $C$  and then finding the lengths  $AC$  and  $BC$ .

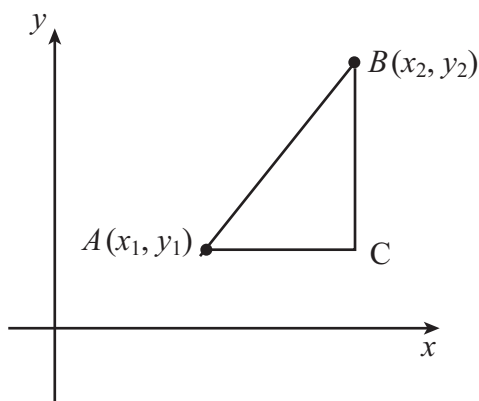


Figure 2.

Here, the point  $C$  is given by  $(x_2, y_1)$ . So the length  $AC$  is  $x_2 - x_1$ , and the length  $BC$  is  $y_2 - y_1$ . Therefore the gradient of  $AB$  is

$$\frac{BC}{AC} = \frac{y_2 - y_1}{x_2 - x_1}.$$

It is a common convention that the gradient of a line is written as  $m$ , so we can write  $m = \frac{y_2 - y_1}{x_2 - x_1}$ .



### Key Point

The gradient,  $m$ , of the line segment joining the points  $A(x_1, y_1)$  and  $B(x_2, y_2)$  is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

We can also think of this another way. Remember that, in the right-angled triangle shown in Figure 3,  $\tan \theta$  equals the opposite over the adjacent, or the change in  $y$  over the change in  $x$  as we move from the beginning to the end of the hypotenuse.

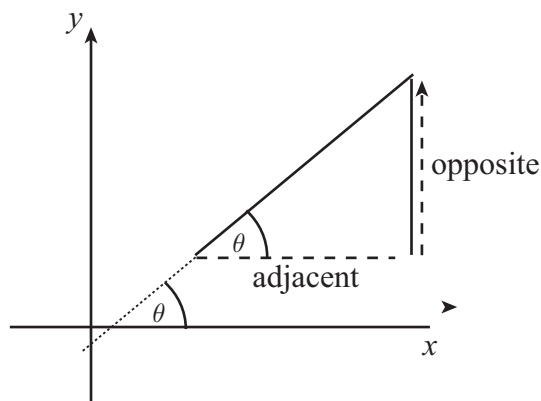


Figure 3.

So the gradient of a line is also the tangent of the angle that the line makes with the horizontal. Note that this is also the tangent of the angle the line makes with the  $x$  axis.



### Key Point

The gradient of a line is equal to the tangent of the angle that the line makes with the horizontal. This is also the tangent of the angle the line makes with the  $x$  axis.

### Example

Suppose  $A$  is the point  $(3, 4)$  and  $B$  is the point  $(8, 14)$ . Then the gradient  $m$  is

$$\begin{aligned} m &= \frac{14 - 4}{8 - 3} \\ &= \frac{10}{5} \\ &= 2. \end{aligned}$$

### Example

Suppose  $A$  is the point  $(0, 4)$  and  $B$  is the point  $(5, 0)$ . Then the gradient  $m$  is

$$\begin{aligned} m &= \frac{0 - 4}{5 - 0} \\ &= -\frac{4}{5}. \end{aligned}$$

Note that the gradient is negative. What does that mean? If we draw a sketch then we can see that the line slopes in a different direction, and the angle made by the line with the  $x$ -axis is obtuse.

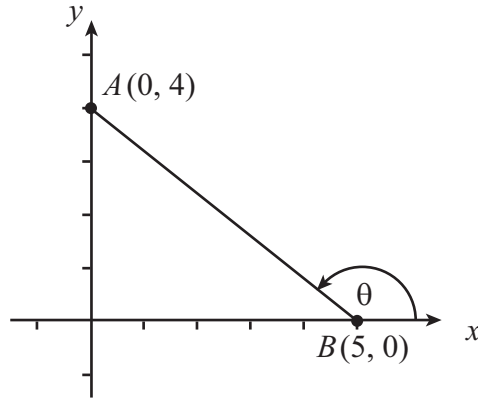


Figure 4.

So we can compare the different cases, where the gradient is positive, negative, or even zero. Take any general line, and let  $\theta$  be the angle it makes with the  $x$ -axis.

When  $\theta$  is acute,  $\tan \theta$  is positive. This is because  $y$  increases as  $x$  increases, so the change in  $y$  and the change in  $x$  are both positive. Therefore the gradient is positive.

When  $\theta$  is obtuse,  $\tan \theta$  is negative. This is because  $y$  decreases as  $x$  increases, so the change in  $y$  and the change in  $x$  have opposite signs. Therefore the gradient is negative.

What about the case where  $\theta = 0$ ? In this case, the line segment is parallel to the  $x$ -axis. Then  $\tan \theta = 0$ , and so gradient is 0.

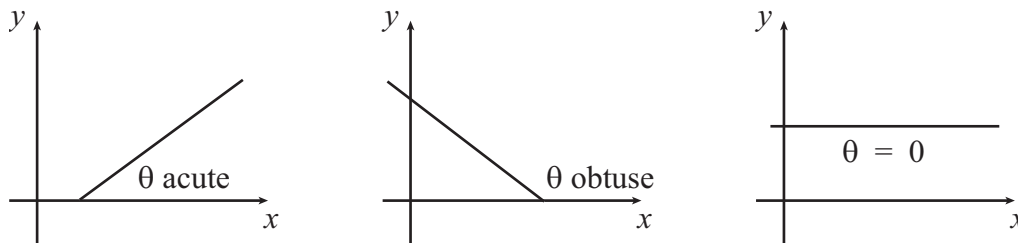


Figure 5.

### Exercise

- Find the gradient of the line segment joining each pair of points:
  - (1, 4) and (5, 7);
  - (2, 0) and (4, 0);
  - (5, -1) and (3, 2)
  - (-1, 2) and (-4, -1);

## 2. Parallel lines

We can apply our knowledge of gradients to the case of parallel lines. If  $l_1$  and  $l_2$  are two parallel lines, then the angles  $\theta_1$  and  $\theta_2$  that they make with the  $x$ -axis are corresponding angles, and so must be equal. Therefore parallel lines must have the same gradient. And lines with the same gradient must be parallel.

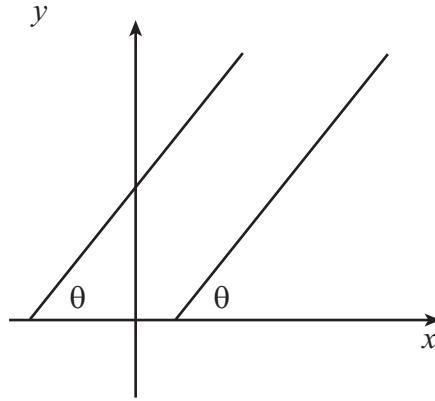


Figure 6.



### Key Point

Suppose that the lines  $l_1, l_2$  have gradients  $m_1$  and  $m_2$ . If the lines are parallel then  $m_1 = m_2$ . We can use this relationship between  $m_1$  and  $m_2$  to decide whether or not two lines are parallel.

### Example

Suppose we take the four points  $A(1, 1)$ ,  $B(4, 2)$ ,  $C(1, -2)$  and  $D(-2, -3)$ . Can we show that  $AB$  is parallel to  $CD$ ?

We shall let  $m_1$  be the gradient of  $AB$ , and let  $m_2$  be the gradient of  $CD$ . Then

$$m_1 = \frac{2 - 1}{4 - 1} = \frac{1}{3}$$

and

$$m_2 = \frac{-3 - (-2)}{-2 - 1} = \frac{-1}{-3} = \frac{1}{3},$$

so the gradients are the same. Therefore the lines  $AB$  and  $CD$  are parallel.

## 3. Perpendicular lines

But what about perpendicular lines? Is there a relationship between their gradients?

If we take the point  $P(a, b)$  and rotate the line  $OP$  anti-clockwise through a right angle about the origin, then  $P$  moves to a new point  $Q$ . We can find the co-ordinates of  $Q$ . To reach  $P$  we went along by  $a$ , and up by  $b$ . So to reach  $Q$  we must go up by  $a$ , and then to the left a distance  $b$ . Therefore  $Q$  must be the point  $(-b, a)$ .

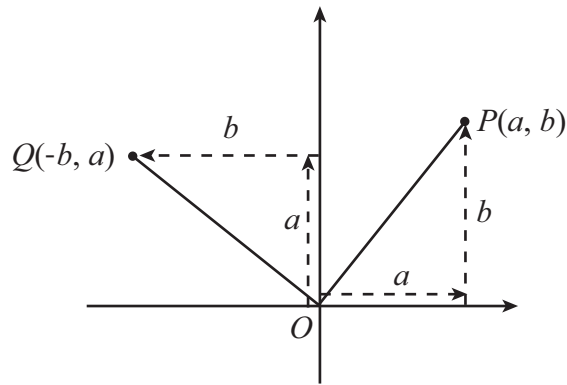


Figure 7.

Now  $OP$  and  $OQ$  are perpendicular lines, and we can calculate their gradients. Let  $m_1$  and  $m_2$  be the gradients of  $OP$  and  $OQ$  respectively. From the formula for gradients,

$$m_1 = \frac{b - 0}{a - 0} = \frac{b}{a}$$

and

$$m_2 = \frac{0 - a}{0 - (-b)} = -\frac{a}{b}.$$

If we multiply the two gradients we get

$$m_1 m_2 = \frac{b}{a} \times -\frac{a}{b} = -1.$$

So when two lines are perpendicular, the product of their gradients is  $-1$ . Conversely, if we know that the product of the gradients of two lines is  $-1$  then the two lines must be perpendicular.



### Key Point

Suppose that the lines  $l_1$ ,  $l_2$  have gradients  $m_1$  and  $m_2$ . If the lines are perpendicular then  $m_1 m_2 = -1$ . Conversely, we can use this relationship between  $m_1$  and  $m_2$  to decide whether or not two lines are perpendicular.

### Example

Suppose we have three points  $A(1, 2)$ ,  $B(3, 4)$  and  $C(0, 3)$ . We can use this formula to show that  $AB$  is perpendicular to  $CA$ .

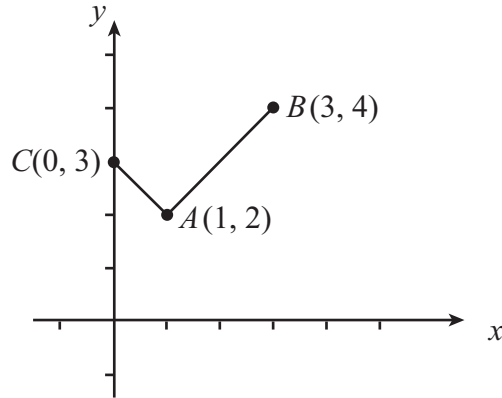


Figure 8.

Let  $m_1$  be the gradient of  $AB$  and  $m_2$  be the gradient of  $CA$ . Then

$$m_1 = \frac{4 - 2}{3 - 1} = \frac{2}{2} = 1$$

and

$$m_2 = \frac{2 - 3}{1 - 0} = -1,$$

so  $m_1 m_2 = 1 \times (-1) = -1$ . Therefore  $AB$  is perpendicular to  $CA$ .

### Exercises

2.

- Show that the line segment joining the points  $(1, 4)$  and  $(3, 10)$  is parallel to the line segment joining the points  $(-5, -10)$  and  $(-2, -1)$ .
- The line segment joining the points  $(4, 1)$  and  $(8, 17)$  is parallel to the line segment joining the points  $(-2, 5)$  and  $(0, a)$ . What is the value of  $a$ ?
- The points  $A$ ,  $B$  and  $C$  have co-ordinates  $(4, 1)$ ,  $(6, -3)$ ,  $(8, -2)$  respectively. Show that triangle  $ABC$  has a right angle at  $B$ .
- The line segment joining the points  $(1, 5)$  and  $(4, 20)$  is perpendicular to the line segment joining  $(0, 3)$  and  $(k, 0)$ . What is the value of  $k$ ?

### Answers

1.

- 0.75, (b) 0, (c) -1.5, (d) 1.

2.

- Both lines have gradient 3 so are parallel.
- $a = 13$ .
- $AB$  has gradient  $-2$ ,  $BC$  has gradient  $0.5$ , the product is  $-1$  so the lines are perpendicular.
- $k = 15$ .