Engineering Maths First Aid Kit

2.22

Sigma notation

Introduction

Sigma notation, $\sum$, provides a concise and convenient way of writing long sums. This leaflet explains how.

1. Sigma notation

The sum

$$1 + 2 + 3 + 4 + 5 + \ldots + 10 + 11 + 12$$

can be written very concisely using the capital Greek letter $\sum$ as

$$\sum_{k=1}^{k=12} k$$

The $\sum$ stands for a sum, in this case the sum of all the values of k as k ranges through all whole numbers from 1 to 12. Note that the lower-most and upper-most values of k are written at the bottom and top of the sigma sign respectively. You may also see this written as $\sum_{k=1}^{k=12} k$, or even as $\sum_{k=1}^{12} k$.

Example
Write out explicitly what is meant by

$$\sum_{k=1}^{k=5} k^3$$

Solution
We must let k range from 1 to 5, cube each value of k, and add the results:

$$\sum_{k=1}^{k=5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3$$

Example
Express $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ concisely using sigma notation.

Solution
Each term takes the form $\frac{1}{k}$ where k varies from 1 to 4. In sigma notation we could write this as

$$\sum_{k=1}^{k=4} \frac{1}{k}$$

Example
The sum

$$x_1 + x_2 + x_3 + x_4 + \ldots + x_{19} + x_{20}$$

can be written

$$\sum_{k=1}^{k=20} x_k$$

There is nothing special about using the letter k. For example

$$\sum_{n=1}^{n=7} n^2 \quad \text{stands for} \quad 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$$

We can also use a little trick to alternate the signs of the numbers between $+$ and $-$. Note that $(-1)^2 = 1$, $(-1)^3 = -1$ and so on.

Example
Write out fully what is meant by

$$\sum_{i=0}^{5} \frac{(-1)^{i+1}}{2i+1}$$

Solution

$$\sum_{i=0}^{5} \frac{(-1)^{i+1}}{2i+1} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11}$$

Exercises

1. Write out fully what is meant by

   a) $\sum_{i=1}^{i=5} i^2$
   b) $\sum_{k=1}^{4}(2k+1)^2$
   c) $\sum_{k=0}^{4}(2k+1)^2$

2. Write out fully what is meant by

$$\sum_{k=1}^{k=3}(\bar{x} - x_k)$$

3. Sigma notation is often used in statistical calculations. For example the **mean**, $\bar{x}$, of the n quantities x_1, $x_2 \ldots$ and x_n, is found by adding them up and dividing the result by n. Show that the mean can be written as

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

4. Write out fully what is meant by $\sum_{i=1}^{4} \frac{i}{i+1}$.

5. Write out fully what is meant by $\sum_{k=1}^{3} \frac{(-1)^k}{k}$.

Answers
1. a) $1^2 + 2^2 + 3^2 + 4^2 + 5^2$, b) $3^2 + 5^2 + 7^2 + 9^2$, c) $1^2 + 3^2 + 5^2 + 7^2 + 9^2$.
2. $(\bar{x} - x_1) + (\bar{x} - x_2) + (\bar{x} - x_3)$, 4. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$, 5. $\frac{-1}{1} + \frac{1}{2} + \frac{-1}{3}$ which equals $-1 + \frac{1}{2} - \frac{1}{3}$.

