Engineering Maths First Aid Kit

7.1

What is a complex number ?

Introduction.

This leaflet explains how the set of real numbers with which you are already familiar is enlarged to include further numbers called **imaginary numbers**. This leads to a study of **complex numbers** which are useful in a variety of engineering applications, especially alternating current circuit analysis.

1. Finding the square root of a negative number.

It is impossible to find the square root of a negative number such as -16. If you try to find this on your calculator you will probably obtain an error message. Nevertheless it becomes useful to construct a way in which we can write down square roots of negative numbers.

We start by introducing a symbol to stand for the square root of -1. Conventionally this symbol is j. That is $j = \sqrt{-1}$. It follows that $j^2 = -1$. Using real numbers we cannot find the square root of a negative number, and so the quantity j is not real. We say it is **imaginary**.

$$j \text{ is an imaginary number such that } j^2 = -1$$

Even though j is not real, using it we can formally write down the square roots of any negative number as shown in the following example.

Example
Write down expressions for the square roots of a) 9, b) -9.

Solution
a) $\sqrt{9} = \pm 3$.

b) Noting that $-9 = 9 \times -1$ we can write

$$\begin{aligned} \sqrt{-9} &= \sqrt{9 \times -1} \\ &= \sqrt{9} \times \sqrt{-1} \\ &= \pm 3 \times \sqrt{-1} \end{aligned}$$

Then using the fact that $\sqrt{-1} = j$ we have

$$\sqrt{-9} = \pm 3j$$

Example
Use the fact that $j^2 = -1$ to simplify a) j^3, b) j^4.

Solution
a) $j^3 = j^2 \times j$. But $j^2 = -1$ and so $j^3 = -1 \times j = -j$.

b) $j^4 = j^2 \times j^2 = (-1) \times (-1) = 1$.

Using the imaginary number j it is possible to solve all quadratic equations.

Example
Use the formula for solving a quadratic equation to solve $2x^2 + x + 1 = 0$.

Solution
We use the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. With $a = 2$, $b = 1$ and $c = 1$ we find

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1^2 - (4)(2)(1)}}{2(2)} \\ &= \frac{-1 \pm \sqrt{-7}}{4} \\ &= \frac{-1 \pm \sqrt{7}j}{4} \\ &= -\frac{1}{4} \pm \frac{\sqrt{7}}{4}j \end{aligned}$$

Exercises

1. Simplify a) $-j^2$, b) $(-j)^2$, c) $(-j)^3$, d) $-j^3$.
2. Solve the quadratic equation $3x^2 + 5x + 3 = 0$.

Answers
1. a) 1, b) -1, c) j, d) j. 2. $-\frac{5}{6} \pm \frac{\sqrt{11}}{6}j$.

2. Complex numbers.

In the previous example we found that the solutions of $2x^2 + x + 1 = 0$ were $-\frac{1}{4} \pm \frac{\sqrt{7}}{4}j$. These are **complex numbers**. A complex number such as $-\frac{1}{4} + \frac{\sqrt{7}}{4}j$ is made up of two parts, a **real part**, $-\frac{1}{4}$, and an **imaginary part**, $\frac{\sqrt{7}}{4}$. We often use the letter z to stand for a complex number and write $z = a + bj$, where a is the real part and b is the imaginary part.

$$z = a + bj$$

where a is the real part and b is the imaginary part of the complex number.

Exercises
1. State the real and imaginary parts of: a) $13 - 5j$, b) $1 - 0.35j$, c) $\cos\theta + j\sin\theta$.

Answers
1. a) real part 13, imaginary part -5, b) $1, -0.35$, c) $\cos\theta$, $\sin\theta$.

