Integration by parts

Introduction

The technique known as **integration by parts** is used to integrate a product of two functions, for example

$$\int e^{2x} \sin 3x \, \mathrm{d}x \qquad \text{and} \qquad \int_0^1 x^3 e^{-2x} \, \mathrm{d}x$$

This leaflet explains how to apply this technique.

1. The integration by parts formula

We need to make use of the integration by parts formula which states:

$$\boxed{\int u \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right) \mathrm{d}x = uv - \int v \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) \mathrm{d}x}$$

Note that the formula replaces one integral, the one on the left, with a different integral, that on the right. The intention is that the latter is simpler to evaluate. Note also that to apply the formula we must let one function in the product equal u. We must be able to differentiate this function to find $\frac{\mathrm{d}u}{\mathrm{d}x}$. We let the other function in the product equal $\frac{\mathrm{d}v}{\mathrm{d}x}$. We must be able to integrate this function, to find v. Consider the the following example:

Example
Find $\int 3x \sin x \, \mathrm{d}x$.

Solution
Compare the required integral with the formula for integration by parts: we see that it makes sense to choose

$$u = 3x \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$$

It follows that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 3 \qquad \text{and} \qquad v = \int \sin x \, \mathrm{d}x = -\cos x$$

(When integrating $\frac{\mathrm{d}v}{\mathrm{d}x}$ to find v there is no need to include a constant of integration. When you become confident with the method, you may like to think about why this is the case.) Applying the formula we obtain

$$\begin{aligned}\int 3x\sin x\,\mathrm{d}x &= uv - \int v\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x \\ &= 3x(-\cos x) - \int(-\cos x).(3)\,\mathrm{d}x \\ &= -3x\cos x + 3\int \cos x\,\mathrm{d}x \\ &= -3x\cos x + 3\sin x + c\end{aligned}$$

2. Dealing with definite integrals

When dealing with definite integrals (those with limits of integration) the corresponding formula is

$$\boxed{\int_a^b u\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\mathrm{d}x = [uv]_a^b - \int_a^b v\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x}$$

Example

Find $\displaystyle\int_0^2 x\mathrm{e}^x\mathrm{d}x$.

Solution

We let $u = x$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$. Then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $v = \mathrm{e}^x$. Using the formula for integration by parts we obtain

$$\begin{aligned}\int_0^2 x\mathrm{e}^x\mathrm{d}x &= [x\mathrm{e}^x]_0^2 - \int_0^2 \mathrm{e}^x.1\mathrm{d}x \\ &= (2\mathrm{e}^2) - (0\mathrm{e}^0) - [\mathrm{e}^x]_0^2 \\ &= 2\mathrm{e}^2 - [\mathrm{e}^2 - 1] \\ &= \mathrm{e}^2 + 1 \qquad \text{(or 8.389 to 3d.p.)}\end{aligned}$$

Exercises

1. Find a) $\int x\sin(2x)\mathrm{d}x$, b) $\int t\mathrm{e}^{3t}\mathrm{d}t$, c) $\int x\cos x\,\mathrm{d}x$.

2. Evaluate the following definite integrals:

a) $\int_0^1 x\cos 2x\,\mathrm{d}x$, b) $\int_0^{\pi/2} x\sin 2x\,\mathrm{d}x$, c) $\int_{-1}^1 t\mathrm{e}^{2t}\mathrm{d}t$.

(Remember to set your calculator to radian mode for evaluating the trigonometric functions.)

3. Find $\int_0^2 x^2\mathrm{e}^x\mathrm{d}x$. (You will need to apply the integration by parts formula twice.)

Answers

1. a) $\frac{\sin 2x}{4} - \frac{x\cos 2x}{2} + c$, b) $\mathrm{e}^{3t}(\frac{t}{3} - \frac{1}{9}) + c$, c) $\cos x + x\sin x + c$.

2. a) 0.1006, b) 0.7854, c) 1.9488

3. 12.778 (3dp).

