Solving Differential Equations with Integrating Factors

mccp-dobson-0111

Introduction

Suppose we have the first order differential equation

$$\frac{dy}{dx} + Py = Q$$

where P and Q are functions involving x only. For example

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^3} \qquad \text{or} \qquad \frac{dy}{dx} - \frac{3y}{x+1} = (x+1)^4.$$

We can solve these differential equations using the technique of an **integrating factor.**

Integrating Factor

We multiply both sides of the differential equation by the integrating factor I which is defined as

$$I = e^{\int P\,dx}.$$

General Solution

Multiplying our original differential equation by I we get that

$$\begin{aligned}\frac{dy}{dx} + Py = Q &\Leftrightarrow I\frac{dy}{dx} + IPy = IQ\\ &\Leftrightarrow \int (I\frac{dy}{dx} + IPy)\,dx = \int IQ\,dx\\ &\Leftrightarrow Iy = \int IQ\,dx \qquad \text{since } \frac{d}{dx}(Iy) = I\frac{dy}{dx} + IPy \text{ by the product rule.}\end{aligned}$$

As both I and Q are functions involving only x in most of the problems you are likely to meet, $\int IQ\,dx$ can usually be found. So the general solution to the differential equation is found by integrating IQ and then re-arranging the formula to make y the subject.

Example

To find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^3}$$

©Katy Dobson
University of Leeds

Alan Slomson
University of Leeds

we first find the integrating factor

$$I = e^{\int P\,dx} = e^{\int \frac{3}{x}\,dx}$$

$$\text{now} \quad \int \frac{3}{x}\,dx = 3\ln x = \ln x^3$$

$$\text{hence} \quad I = e^{\ln x^3} = x^3.$$

Then we multiply the differential equation by I to get

$$x^3\frac{dy}{dx} + 3x^2y = e^x$$

so integrating both sides we have $x^3y = e^x + c$ where c is a constant. Thus the general solution is

$$y = \frac{e^x + c}{x^3}.$$

Example

To find the general solution of the differential equation

$$\frac{dy}{dx} - \frac{3y}{x+1} = (x+1)^4$$

we first find the integrating factor

$$I = e^{\int P\,dx} = e^{\int \frac{-3}{x+1}\,dx}$$

$$\text{now} \quad \int \frac{-3}{x+1}\,dx = -3\ln(x+1) = \ln(x+1)^{-3}$$

$$\text{hence} \quad I = e^{\ln(x+1)^{-3}} = (x+1)^{-3} = \frac{1}{(x+1)^3}.$$

Then multiplying the differential equation by I we get

$$\frac{1}{(x+1)^3}\frac{dy}{dx} - \frac{3y}{(x+1)^4} = (x+1)$$

so integrating both sides we have

$$\frac{y}{(x+1)^3} = \frac{1}{2}x^2 + x + c \quad \text{where } c \text{ is a constant.}$$

Thus the general solution is

$$y = (x+1)^3(\frac{1}{2}x^2 + x + c).$$

Exercises

Find the general solution of

1. $\dfrac{dy}{dx} + \dfrac{2y}{x} = \dfrac{\sin x}{x^2}$ 2. $\dfrac{dy}{dx} - \dfrac{y}{x} = -xe^{-x}$ 3. $\dfrac{dy}{dx} + 2xy = x$ 4. $\dfrac{dy}{dx} - \dfrac{2y}{x} = 3x^3$

Answers

1. $y = \dfrac{c - \cos x}{x^2}$ 2. $y = x(e^{-x} + c)$ 3. $y = \dfrac{1}{2} + ce^{-x^2}$ 4. $y = \dfrac{3}{2}x^4 + cx^2$

Alan Slomson
University of Leeds