Proof by Induction : Further Examples

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Example

Prove by induction that $11^n - 6$ is divisible by 5 for every positive integer n.

Solution

Let $P(n)$ be the mathematical statement

$$11^n - 6 \text{ is divisible by } 5.$$

Base Case: When $n = 1$ we have $11^1 - 6 = 5$ which is divisible by 5. So $P(1)$ is correct.

Induction hypothesis: Assume that $P(k)$ is correct for some positive integer k. That means $11^k - 6$ is divisible by 5 and hence $11^k - 6 = 5m$ for some integer m. So $11^k = 5m + 6$.

Induction step: We will now show that $P(k + 1)$ is correct. Always keep in mind what we are aiming for and what we know to be true. In this case we want to show that $11^{k+1} - 6$ can be expressed as a multiple of 5, so we will start with the formula $11^{k+1} - 6$ and we will rearrange it into something involving multiples of 5. At some point we will also want to use the assumption that $11^k = 5m + 6$.

$$\begin{aligned}
11^{k+1} - 6 &= (11 \times 11^k) - 6 && \text{by the laws of powers} \\
&= 11(5m + 6) - 6 && \text{by the induction hypothesis} \\
&= 11(5m) + 66 - 6 && \text{by expanding the bracket} \\
&= 5(11m) + 60 && \\
&= 5(11m + 12) && \text{since both parts of the formula have a common factor of 5.}
\end{aligned}$$

As $11m + 12$ is an integer we have that $11^{k+1} - 6$ is divisible by 5, so $P(k + 1)$ is correct. Hence by mathematical induction $P(n)$ is correct for all positive integers n.

Example

Prove by induction that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for every positive integer n.

Solution

Let $P(n)$ be the statement $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Base Case: When $n = 1$ the left hand side of the equation is 1 and the right hand side is $\frac{1(1+1)(2+1)}{6} = \frac{2\times 3}{6} = 1$. So $P(1)$ is correct.

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Induction hypothesis: Assume that $P(k)$ is correct for some positive integer k. That means that the left hand side of the equation equals the right hand side, so $1^2+2^2+3^2+\ldots+k^2 = \frac{k(k+1)(2k+1)}{6}$.

Induction step: We will now show that $P(k+1)$ is correct. Keep in mind what we are aiming for, so in this case the right hand side of the equation should be $\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$. So starting with the left hand side we have

$$
\begin{aligned}
1^2+2^2+3^2+\ldots+k^2+(k+1)^2 = \\
&= (1^2+2^2+3^2+\ldots+k^2)+(k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \quad \text{by the induction hypothesis} \\
&= \frac{k(k+1)(2k+1)}{6}+\frac{6(k+1)^2}{6} \quad \text{by making each part a fraction over 6} \\
&= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} \quad \text{by making it a single fraction over 6} \\
&= \frac{(k+1)[k(2k+1)+6(k+1)]}{6} \quad \text{by taking out the common factor} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \quad \text{by expanding out the square brackets} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \quad \text{by factorizing} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \quad \text{this is the right hand side.}
\end{aligned}
$$

So $P(k+1)$ is correct. Hence by mathematical induction $P(n)$ is correct for all positive integers n.

Example

Prove by induction that $2^n > 2n$ for every positive integer $n > 2$.

Solution

Let $P(n)$ be the mathematical statement

$$2^n > 2n.$$

Base Case: When $n=3$ we have $2^3 = 8 > 6 = 2\times 3$. So $P(3)$ is correct.
Induction hypothesis: Assume that $P(k)$ is correct for some positive integer k. That means that $2^k > 2k$.
Induction step: We will now show that $P(k+1)$ is correct.

$$
\begin{aligned}
2^{k+1} = 2\times 2^k > 2\times 2k \quad &\text{by the induction hypothesis} \\
= 2(k+1).&
\end{aligned}
$$

So $P(k+1)$ is correct. Hence by mathematical induction $P(n)$ is correct for all positive integers $n > 2$.

Exercises

Prove by induction that

1. $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 3}+\ldots+\frac{1}{n\times(n+1)} = \frac{n}{n+1}$ for all positive integers.
2. n^3-n is divisible by 6 for all positive integers.
3. $2^{n+2}+3^{2n+1}$ is divisible by 7 for all positive integers.

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