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in mathematics & statistics support

The modulus and argument of a complex number

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In this unit you are going to learn about the **modulus** and **argument** of a complex number. These are quantities which can be recognised by looking at an Argand diagram. Recall that any complex number, z, can be represented by a point in the complex plane as shown in Figure 1.

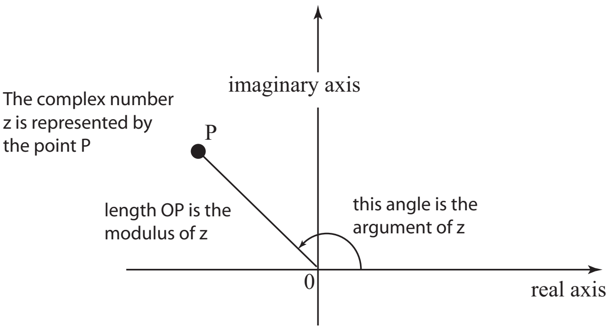

Figure 1. The complex number z is represented by point P. Its modulus and argument are shown.

We can join point P to the origin with a line segment, as shown. We associate with this line segment two important quantities. The length of the line segment, that is OP, is called the **modulus** of the complex number. The angle from the positive axis to the line segment is called the **argument** of the complex number, z.

The modulus and argument are fairly simple to calculate using trigonometry.

Example. Find the modulus and argument of $z = 4 + 3\text{i}$.

Solution. The complex number $z = 4 + 3\text{i}$ is shown in Figure 2. It has been represented by the point Q which has coordinates $(4, 3)$. The modulus of z is the length of the line OQ which we can find using Pythagoras' theorem.

$$(OQ)^2 = 4^2 + 3^2 = 16 + 9 = 25$$

and hence $OQ = 5$.

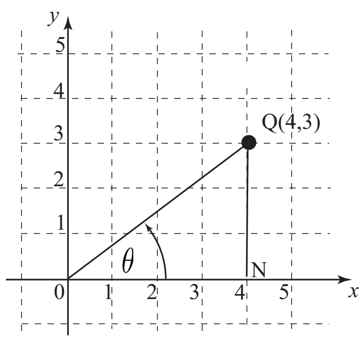

Figure 2. The complex number $z = 4 + 3\text{i}$.

Hence the modulus of $z = 4 + 3\text{i}$ is 5. To find the argument we must calculate the angle between the x axis and the line segment OQ. We have labelled this θ in Figure 2.

By referring to the right-angled triangle OQN in Figure 2 we see that

$$\tan\theta = \frac{3}{4}$$

$$\theta = \tan^{-1}\frac{3}{4} = 36.97°$$

To summarise, the modulus of $z = 4 + 3\text{i}$ is 5 and its argument is $\theta = 36.97°$. There is a special symbol for the modulus of z; this is $|z|$. So, in this example, $|z| = 5$. We also have an abbreviation for argument: we write arg$(z) = 36.97°$.

When the complex number lies in the first quadrant, calculation of the modulus and argument is straightforward. For complex numbers outside the first quadrant we need to be a little bit more careful. Consider the following example.

Example.

Find the modulus and argument of $z = 3 - 2\text{i}$.

Solution. The Argand diagram is shown in Figure 3. The point P with coordinates $(3, -2)$ represents $z = 3 - 2\text{i}$.

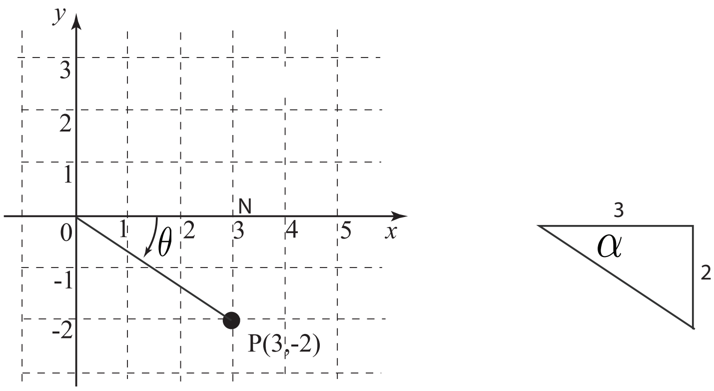

Figure 3. The complex number $z = 3 - 2\text{i}$.

We use Pythagoras' theorem in triangle ONP to find the modulus of z:

$$(OP)^2 = 3^2 + 2^2 = 13$$

$$OP = \sqrt{13}$$

Using the symbol for modulus, we see that in this example $|z| = \sqrt{13}$.

We must be more careful with the argument. When the angle θ shown in Figure 3 is measured in a clockwise sense convention dictates that the angle is negative. We can find the size of the angle by referring to the right-angled triangle shown. In that triangle $\tan\alpha = \frac{2}{3}$ so that $\alpha = \tan^{-1}\frac{2}{3} = 33.67°$. This is not the argument of z. The argument of z is $\theta = -33.67°$. We often write this as arg$(z) = -33.67°$.

In the next unit we show how the modulus and argument are used to define the **polar form** of a complex number.