mathcentre

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Facts & Formulae

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Algebra

$$(x+k)(x-k) = x^2 - k^2$$
$$(x+k)^2 = x^2 + 2kx + k^2, \quad (x-k)^2 = x^2 - 2kx + k^2$$
$$x^3 \pm k^3 = (x \pm k)(x^2 \mp kx + k^2)$$

Formula for solving a quadratic equation:

$$\text{if } ax^2 + bx + c = 0 \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Laws of Indices

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{mn}$$
$$a^0 = 1 \qquad a^{-m} = \frac{1}{a^m} \qquad a^{1/n} = \sqrt[n]{a} \qquad a^{\frac{m}{n}} = (\sqrt[n]{a})^m$$

Laws of Logarithms

For any positive base b (with $b \neq 1$)

$$\log_b A = c \quad \text{means} \quad A = b^c$$
$$\log_b A + \log_b B = \log_b AB, \qquad \log_b A - \log_b B = \log_b \frac{A}{B},$$
$$n \log_b A = \log_b A^n, \qquad \log_b 1 = 0, \qquad \log_b b = 1$$

Formula for change of base: $\log_a x = \frac{\log_b x}{\log_b a}$

Logarithms to base e, denoted $\log_e$ or alternatively ln are called *natural logarithms*. The letter e stands for the exponential constant which is approximately 2.718.

Partial fractions

For *proper fractions* $\frac{P(x)}{Q(x)}$ where P and Q are polynomials with the degree of P less than the degree of Q:

a *linear factor* $ax+b$ in the denominator produces a partial fraction of the form $\frac{A}{ax+b}$

repeated linear factors $(ax+b)^2$ in the denominator produce partial fractions of the form $\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$

a *quadratic factor* ax^2+bx+c in the denominator produces a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$

Improper fractions require an additional term which is a polynomial of degree $n-d$ where n is the degree of the numerator and d is the degree of the denominator.

Inequalities:

$a > b$ means a is greater than b

$a < b$ means a is less than b

$a \geqslant b$ means a is greater than or equal to b

$a \leqslant b$ means a is less than or equal to b

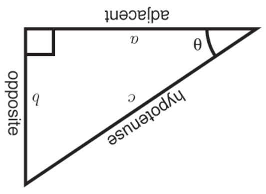
Trigonometry

Degrees and radians

$$360° = 2\pi \text{ radians}, \quad 1° = \frac{2\pi}{360} = \frac{\pi}{180} \text{ radians}$$
$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \approx 57.3°$$

Trig ratios for an acute angle θ:

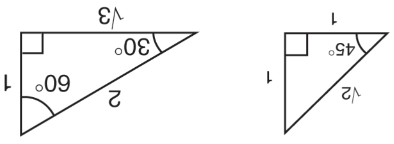
$$\sin\theta = \frac{\text{side opposite to }\theta}{\text{hypotenuse}} = \frac{b}{c}$$
$$\cos\theta = \frac{\text{side adjacent to }\theta}{\text{hypotenuse}} = \frac{a}{c}$$
$$\tan\theta = \frac{\text{side opposite to }\theta}{\text{side adjacent to }\theta} = \frac{b}{a}$$



Pythagoras' theorem

$$a^2 + b^2 = c^2$$

Standard triangles:



$$\sin 45° = \frac{1}{\sqrt{2}}, \quad \cos 45° = \frac{1}{\sqrt{2}}, \quad \tan 45° = 1$$
$$\sin 30° = \frac{1}{2}, \quad \cos 30° = \frac{\sqrt{3}}{2}, \quad \tan 30° = \frac{1}{\sqrt{3}}$$
$$\sin 60° = \frac{\sqrt{3}}{2}, \quad \cos 60° = \frac{1}{2}, \quad \tan 60° = \sqrt{3}$$

Common trigonometric identities

$\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\cos B = \cos(A-B) + \cos(A+B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\sin^2 A + \cos^2 A = 1$

$1 + \cot^2 A = \text{cosec}^2 A, \quad \tan^2 A + 1 = \sec^2 A$

$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$\sin 2A = 2\sin A\cos A$

$\sin^2 A = \frac{1-\cos 2A}{2}, \qquad \cos^2 A = \frac{1+\cos 2A}{2}$

$\sin^2 A$ is the notation used for $(\sin A)^2$. Similarly $\cos^2 A$ means $(\cos A)^2$ etc. This notation is used with trigonometric and hyperbolic functions but with positive integer powers only.

Hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$
$$\text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic identities

$e^x = \cosh x + \sinh x, \quad e^{-x} = \cosh x - \sinh x$

$\cosh^2 x - \sinh^2 x = 1$

$1 - \tanh^2 x = \text{sech}^2 x$

$\coth^2 x - 1 = \text{cosech}^2 x$

$\sinh(x \pm y) = \sinh x\cosh y \pm \cosh x \sinh y$

$\cosh(x \pm y) = \cosh x\cosh y \pm \sinh x \sinh y$

$\sinh 2x = 2\sinh x\cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^2 x = \frac{\cosh 2x + 1}{2}$

$\sinh^2 x = \frac{\cosh 2x - 1}{2}$

Inverse hyperbolic functions

$\cosh^{-1} x = \ln(x + \sqrt{x^2-1}) \quad$ for $x \geqslant 1$

$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad$ for $-1 < x < 1$

The Greek alphabet

A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ϵ	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

Written by Tony Croft and Geoff Simpson for the Mathematics Learning Support Centre at Loughborough University
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Differentiation

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
k, constant	0
x^n, any constant n	nx^{n-1}
e^x	e^x
$\ln x = \log_e x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x = \frac{\sin x}{\cos x}$	$\sec^2 x$
$\text{cosec}\, x = \frac{1}{\sin x}$	$-\text{cosec}\, x \cot x$
$\sec x = \frac{1}{\cos x}$	$\sec x \tan x$
$\cot x = \frac{\cos x}{\sin x}$	$-\text{cosec}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$
$\tanh x$	$\text{sech}^2 x$
$\text{sech}\, x$	$-\text{sech}\, x \tanh x$
$\text{cosech}\, x$	$-\text{cosech}\, x \coth x$
$\coth x$	$-\text{cosech}^2 x$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

The linearity rule for differentiation

$$\frac{d}{dx}(au+bv) = a\frac{du}{dx} + b\frac{dv}{dx} \qquad a, b \text{ constant}$$

The product and quotient rules for differentiation

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

The chain rule for differentiation

If $y = y(u)$ where $u = u(x)$ then $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

For example,
if $y = (\cos x)^{-1}$, $\frac{dy}{dx} = -1(\cos x)^{-2}(-\sin x)$

Integration

$f(x)$	$\int f(x)\,dx = F(x) + c$	
k, constant	$kx + c$	
x^n, $(n \neq -1)$	$\frac{x^{n+1}}{n+1} + c$	
$x^{-1} = \frac{1}{x}$	$\ln x + c$	$x > 0$
	$\ln(-x) + c$	$x < 0$
e^x	$e^x + c$	
$\cos x$	$\sin x + c$	
$\sin x$	$-\cos x + c$	
$\tan x$	$\ln(\sec x) + c$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\sec x$	$\ln(\sec x + \tan x) + c$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\text{cosec}\, x$	$\ln(\text{cosec}\, x - \cot x) + c$	$0 < x < \pi$
$\cot x$	$\ln(\sin x) + c$	$0 < x < \pi$
$\cosh x$	$\sinh x + c$	
$\sinh x$	$\cosh x + c$	
$\tanh x$	$\ln\cosh x + c$	
$\coth x$	$\ln\sinh x + c$	$x > 0$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + c$	$a > 0$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\frac{x-a}{x+a} + c$	$\lvert x\rvert > a > 0$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\frac{a+x}{a-x} + c$	$\lvert x\rvert < a$
$\frac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}\frac{x}{a} + c$	$a > 0$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\frac{x}{a} + c$	$x \geqslant a > 0$
$\frac{1}{\sqrt{x^2+k}}$	$\ln(x + \sqrt{x^2+k}) + c$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a} + c$	$-a \leqslant x \leqslant a$
$f(ax+b)$	$\frac{1}{a}F(ax+b) + c$	$a \neq 0$
e.g. $\cos(2x-3)$	$\frac{1}{2}\sin(2x-3) + c$	

The linearity rule for integration

$$\int (af(x) + bg(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx, \quad (a, b \text{ constant})$$

Integration by substitution

$$\int f(u)\frac{du}{dx}dx = \int f(u)du \quad \text{and} \quad \int_a^b f(u)\frac{du}{dx}dx = \int_{u(a)}^{u(b)} f(u)du$$

Integration by parts

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b \frac{du}{dx}v\,dx$$

Alternative form:

$$\int_a^b f(x)g(x)\,dx = \left[f(x)\int g(x)dx\right]_a^b - \int_a^b \frac{df}{dx}\left\{\int g(x)dx\right\}dx$$

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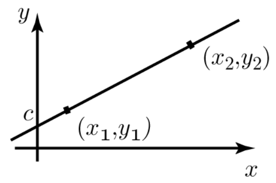
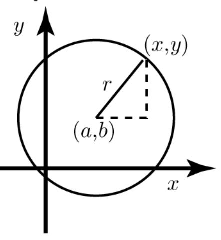
Graphs of common functions

Linear $y = mx + c$, m=gradient, c = vertical intercept



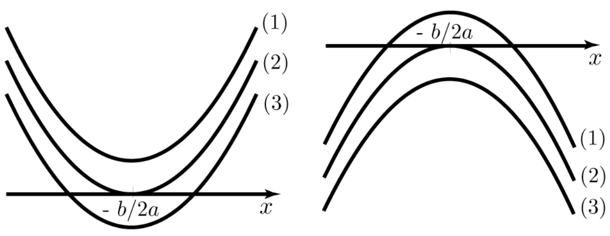
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The equation of a circle centre (a,b), radius r



$$(x-a)^2 + (y-b)^2 = r^2$$

Quadratic functions $y = ax^2 + bx + c$



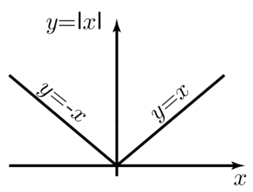
$a > 0$	$a < 0$
(1) $b^2 - 4ac < 0$	(1) $b^2 - 4ac > 0$
(2) $b^2 - 4ac = 0$	(2) $b^2 - 4ac = 0$
(3) $b^2 - 4ac > 0$	(3) $b^2 - 4ac < 0$

Completing the square

$$\text{If } a \neq 0, \quad ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

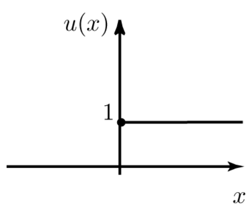
The modulus function

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

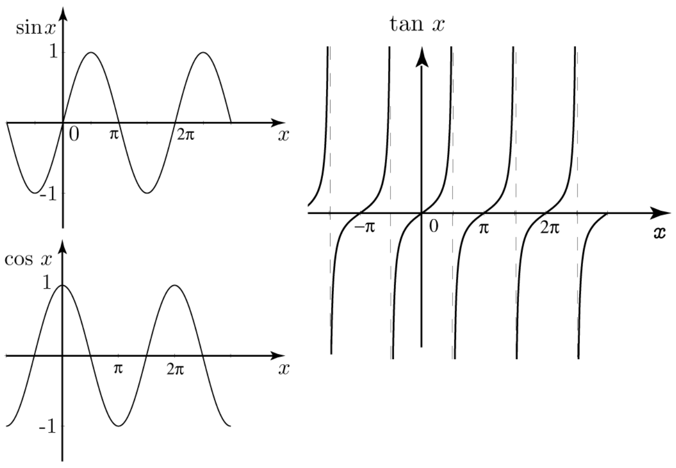


The unit step function, $u(x)$

$$u(x) = \begin{cases} 1 & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$$

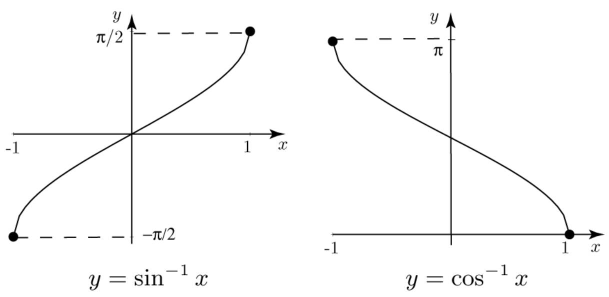


Trigonometric functions

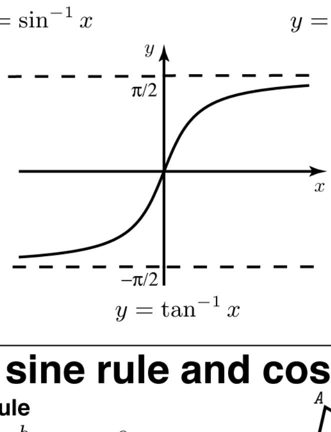


The sine and cosine functions are periodic with period 2π. The tangent function is periodic with period π.

Inverse trigonometric functions



$y = \sin^{-1} x$ $\quad$ $y = \cos^{-1} x$



$y = \tan^{-1} x$

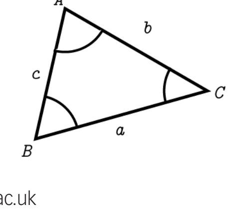
The sine rule and cosine rule

The sine rule

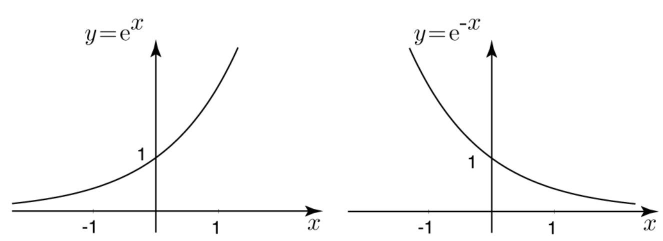
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The cosine rule

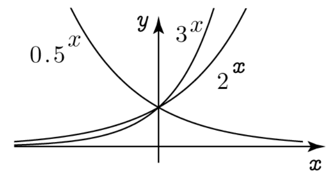
$$a^2 = b^2 + c^2 - 2bc\cos A$$



Exponential functions

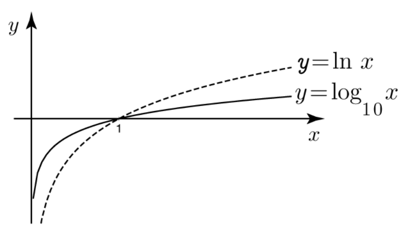


Graph of $y = e^x$ showing exponential growth $\quad$ Graph of $y = e^{-x}$ showing exponential decay



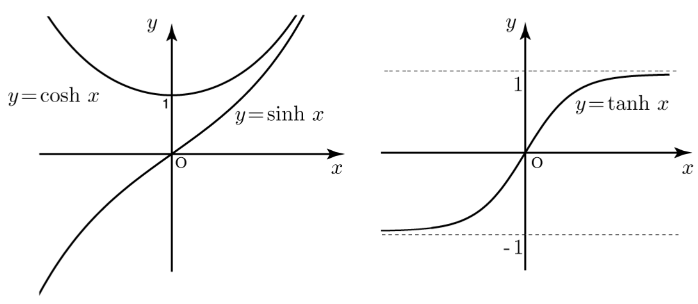
Graphs of $y = 0.5^x$, $y = 3^x$, and $y = 2^x$

Logarithmic functions



Graphs of $y = \ln x$ and $y = \log_{10} x$

Hyperbolic functions

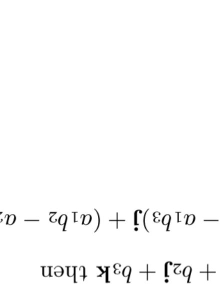


Graphs of $y = \sinh x$, $y = \cosh x$ and $y = \tanh x$

Complex Numbers

Cartesian form: $z = a + bj$
where $j = \sqrt{-1}$

Polar form:
$z = r(\cos\theta + j\sin\theta) = r\angle\theta$
$a = r\cos\theta, \quad b = r\sin\theta,$
$\tan\theta = \frac{b}{a}$



Exponential form: $z = re^{j\theta}$

Euler's relations
$e^{j\theta} = \cos\theta + j\sin\theta, \quad e^{-j\theta} = \cos\theta - j\sin\theta$

Multiplication and division in polar form

$$z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2), \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

If $z = r\angle\theta$, then $z^n = r^n\angle(n\theta)$

De Moivre's theorem

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta$$

Relationship between hyperbolic and trig functions
$\cos jx = \cosh x, \quad \sin jx = j\sinh x$
$\cosh jx = \cos x, \quad \sinh jx = j\sin x$
i rather than j may be used to denote $\sqrt{-1}$.

Vectors

If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$

Scalar product

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Vector product

$\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\hat{\mathbf{e}}$

$\hat{\mathbf{e}}$ is a unit vector perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$ in a sense defined by the right hand screw rule.

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Sequences and Series

Arithmetic progression: $a, a+d, a+2d, \ldots$
a = first term, d = common difference,
kth term $= a + (k-1)d$
Sum of n terms, $S_n = \frac{n}{2}(2a + (n-1)d)$

Sum of the first n **integers,**
$1 + 2 + 3 + \ldots + n =$

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$$

Sum of the squares of the first n **integers,**
$1^2 + 2^2 + 3^2 + \ldots + n^2 =$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

Geometric progression: $a, ar, ar^2, \ldots$
a = first term, r = common ratio,
kth term $= ar^{k-1}$
Sum of n terms, $S_n = \frac{a(1-r^n)}{1-r}$, provided $r \neq 1$

Sum of an infinite geometric series:
$S_\infty = \frac{a}{1-r}, \quad -1 < r < 1$

The binomial theorem
If n is a positive integer

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n$$

When n is negative or fractional, the series is infinite and converges when $-1 < x < 1$

Standard power series expansions

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \text{ for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \text{ for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \text{ for all } x$$

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \text{ for } -1 < x \leqslant 1 \text{ only}$$

The exponential function as the limit of a sequence

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$$

Matrices and Determinants

The 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has determinant

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ has determinant

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(expanded along the first row).

The inverse of a 2×2 **matrix**
If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
provided that $ad - bc \neq 0$.

Matrix multiplication: for 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\beta & a\gamma + b\delta \\ c\alpha + d\beta & c\gamma + d\delta \end{pmatrix}$$

Remember that $AB \neq BA$ except in special cases.

The Binomial Coefficients

The coefficient of x^k in the binomial expansion of $(1+x)^n$ when n is a positive integer is denoted by $\binom{n}{k}$ or ${}^nC_k$.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

$0! = 1, \quad n! = n(n-1)!$
so, for example, $4! = 1.2.3.4$

The pattern of the coefficients is seen in

Pascal's triangle:

```
            1     1
         1     2     1
      1     3     3     1
   1     4     6     4     1
1     5    10    10     5     1
⋮     ⋮     ⋮     ⋮     ⋮     ⋮     ⋮
```

${}^nC_k$ is the number of subsets with k elements that can be chosen from a set with n elements.