Integrating algebraic fractions 2

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Sometimes the integral of an algebraic fraction can be found by first expressing the algebraic fraction as the sum of its partial fractions. In this unit we look at the case where the denominator of the fraction involves an irreducible quadratic expression.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- integrate an algebraic fraction where the denominator involves an irreducible quadratic expression.

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1. Introduction

In this unit we are going to look at how we can integrate some more algebraic fractions. We shall concentrate on the case where the denominator of the fraction involves an irreducible quadratic factor. The case where all the factors of the denominator are linear has been covered in the first unit on integrating algebraic fractions.

2. Algebraic fractions with an irreducible quadratic factor

When the denominator of a fraction contains a quadratic, $ax^2 + bx + c$, which will not factorise into two linear factors (said to be an irreducible quadratic factor) the appropriate form of partial fractions is

$$\frac{Ax + B}{ax^2 + bx + c}.$$

Suppose the value of A turns out to be zero. Then we have an integrand of the form

$$\frac{\text{constant}}{ax^2 + bx + c}.$$

A term like this can be integrated by completing the square as in the following example.

Example

Suppose we wish to find $\displaystyle\int \frac{1}{x^2 + x + 1}\mathrm{d}x$.

We start by completing the square in the denominator to give

$$\int \frac{1}{x^2 + x + 1}\mathrm{d}x = \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}\mathrm{d}x\,.$$

If we now substitute $u = x + \frac{1}{2}$ we obtain $\displaystyle\int \frac{1}{u^2 + \frac{3}{4}}\mathrm{d}u$. There is a standard result which we quote that

$$\int \frac{1}{a^2 + x^2}\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\,.$$

This enables us to complete this example, with $a = \sqrt{3}/2$, and obtain $\displaystyle\frac{1}{\sqrt{3}/2}\tan^{-1}\frac{u}{\sqrt{3}/2}$. In terms of the original variable, x, we have

$$\int \frac{1}{x^2 + x + 1}\mathrm{d}x = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x + 1}{\sqrt{3}} + c\,.$$

On the other hand, if A is non-zero it may turn out that the numerator is the derivative of the denominator, or can be easily made so. In such cases, the standard result

$$\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln|f(x)| + c$$

can be used.

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Consider the following example.

Example

Suppose we wish to find $\displaystyle\int \frac{2x+1}{x^2+x+1}\mathrm{d}x$.

Here the derivative of x^2+x+1 is $2x+1$ and so immediately we can write down the answer:

$$\int \frac{2x+1}{x^2+x+1}\mathrm{d}x = \ln|x^2+x+1| + c\,.$$

Finally, we may have values of A and B such that the numerator is not the derivative of the denominator. In such a case we have to adjust the numerator to make it so, and then compensate by including another term as illustrated in the following example.

Example

Suppose we wish to find $\displaystyle\int \frac{x}{x^2+x+1}\mathrm{d}x$.

Again we shall use the result that $\displaystyle\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln|f(x)| + c$ but unfortunately the numerator is not quite equal to the derivative of the denominator. We would like the denominator to be $2x+1$. We can adjust the integrand by writing it as follows:

$$\frac{x}{x^2+x+1} = \frac{1}{2}\left(\frac{2x}{x^2+x+1}\right)\,.$$

This has the effect of introducing the required $2x$ in the numerator. To obtain the require $+1$, we write

$$\frac{1}{2}\left(\frac{2x}{x^2+x+1}\right) = \frac{1}{2}\left(\frac{2x+1 \qquad -1}{x^2+x+1}\right) = \frac{1}{2}\left(\frac{2x+1}{x^2+x+1} - \frac{1}{x^2+x+1}\right)\,.$$

Each of these terms can be integrated using the results of the two previous examples and we obtain

$$\int \frac{x}{x^2+x+1}\mathrm{d}x = \frac{1}{2}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + c\,.$$

So you see that you need a combination of a variety of techniques together with some ingenuity and skill. Practice is essential so that you experience a wide variety of such problems.

Example

Find $\displaystyle\int \frac{1}{x(x^2+1)}\mathrm{d}x$.

Here we express the integrand in partial fractions, noting in particular the appropriate form to use when dealing with an irreducible quadratic factor.

$$\begin{aligned}\frac{1}{x(x^2+1)} &= \frac{A}{x} + \frac{Bx+C}{x^2+1}\\ &= \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}\,.\end{aligned}$$

The fractions on the left and right are equal for all values of x. The denominators are equal, and so too must be the numerators. So, from just the numerators:

$$1 = A(x^2+1) + (Bx+C)x\,.$$

We can substitute some sensible values for x in order to find the numbers A, B and C, or we can equate coefficients. We can also use a combination of these two methods. If we substitute $x = 0$ we obtain

$$1 = A\,.$$

Next, we can compare the coefficients of x^2 on both sides:

$$0 = A + B\,.$$

But we already know that $A = 1$, and so B must equal -1. And finally, we compare the coefficients of x on both sides:

$$0 = C\,.$$

Returning to the integral, and using these values of A, B and C we find

$$\int \frac{1}{x(x^2+1)}\mathrm{d}x \quad = \quad \int \frac{1}{x}\,\mathrm{d}x - \int \frac{x}{x^2+1}\,\mathrm{d}x\,.$$

The first integral on the right is a standard form. The second is adjusted as follows to make the numerator the derivative of the denominator:

$$\int \frac{x}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x}{x^2+1}\,\mathrm{d}x\,.$$

Then

$$\begin{aligned}\int \frac{1}{x(x^2+1)}\mathrm{d}x \quad &= \quad \int \frac{1}{x}\,\mathrm{d}x - \frac{1}{2}\int \frac{2x}{x^2+1}\,\mathrm{d}x \\ &= \quad \ln|x| - \frac{1}{2}\ln|x^2+1| + c\end{aligned}$$

Using the laws of logarithms the first two terms can be combined to express the answer as a single logarithm if required:

$$\begin{aligned}\ln|x| - \frac{1}{2}\ln|x^2+1| \quad &= \quad \ln|x| - \ln|x^2+1|^{1/2} \\ &= \quad \ln|x| - \ln|\sqrt{x^2+1}| \\ &= \quad \ln\left|\frac{x}{\sqrt{x^2+1}}\right|\,.\end{aligned}$$

Exercises

1. Find the following integrals by expressing the integrand in partial fractions.

(a) $\displaystyle\int \frac{3x^2-6x+4}{(x^2+4)(x-3)}\mathrm{d}x$ (b) $\displaystyle\int \frac{6x^2+10x+5}{(x+1)(2x^2+3x+2)}\mathrm{d}x$

2. By completing the square find $\displaystyle\int \frac{1}{2x^2+x+3}\mathrm{d}x$.

3. Find $\displaystyle\int \frac{x+1}{x^2+x+3}\mathrm{d}x$.

4. Express in partial fractions and then integrate $\displaystyle\frac{2x^2+2x+19}{(x-2)(2x^2+6x+11)}$.

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Answers

1. (a) the partial fractions are $\dfrac{1}{x-3}+\dfrac{2x}{x^2+4}$; the integral is $\ln|x-3|+\ln|x^2+4|+C$.

(b) the partial fractions are $\dfrac{1}{x+1}+\dfrac{3+4x}{2x^2+3x+2}$; the integral is $\ln|x+1|+\ln|2x^2+3x+2|+C$.

2. $\dfrac{2}{\sqrt{23}}\tan^{-1}\left(\dfrac{4x+1}{\sqrt{23}}\right)+C$.

3. $\dfrac{1}{2}\ln(x^2+x+3)+\dfrac{1}{\sqrt{11}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{11}}\right)$.

4. $\ln|x-2|-\dfrac{4}{\sqrt{13}}\tan^{-1}\dfrac{2x+3}{\sqrt{13}}+C$.

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