$a\cos x + b\sin x = R\cos(x - \alpha)$

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In this unit we explore how the sum of two trigonometric functions, e.g. $3\cos x + 4\sin x$, can be expressed as a single trigonometric function. Having the ability to do this enables you to solve certain sorts of trigonometric equations and find maximum and minimum values of some trigonometric functions.

In order to master the techniques explained here it is vital that you undertake the practice exercises provided.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- express the sum of two trigonometric functions, $a\cos x + b\sin x$, in the form $R\cos(x - \alpha)$.
- use this technique to solve some equations.
- use this technique to locate maximum and minimum values.

Contents

1. Introduction

In this unit we are going to have a look at a particular form of trigonometric function. Consider the following function, which is a sum of two trigonometric functions:

$$3\cos x + 4\sin x$$

You will find that in some applications, for example in solving trigonometric equations, it is helpful to write these two terms as a single term. We study how this can be achieved in this unit.

2. The graph of $y = 3\cos x + 4\sin x$

We start by having a look at the graph of the function $y = 3\cos x + 4\sin x$. This is illustrated in Figure 1.

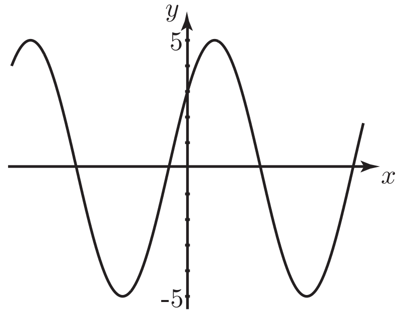

Figure 1. A graph of the function $y = 3\cos x + 4\sin x$.

If you have a graphical calculator you should check that you can reproduce this graph for yourself. (The calculator mode should be set to work in radians rather than degrees). Observe that the maximum and minimum values of this function are 5 and -5 respectively. Also note that the graph looks like the graph of a cosine function except that it is displaced a little to the right.

To emphasise this, in Figure 2 we show this function again, and also the graph of $y = 5\cos x$ for comparison.

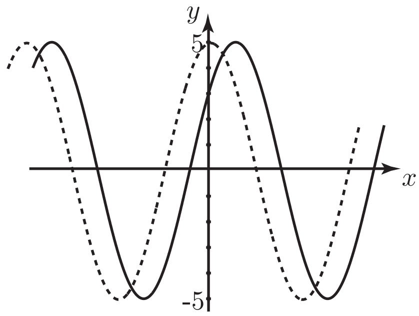

Figure 2. Graphs of $y = 5\cos x$ and $y = 3\cos x + 4\sin x$.

In fact the function $3\cos x + 4\sin x$ can be expressed in the form $5\cos(x - \alpha)$ where α is an angle very close to 1 radian. It is the presence of the term α which causes the horizontal displacement. In the following section we will see how the more general expression $a\cos x + b\sin x$ can be written as $R\cos(x - \alpha)$. In the example above note that the three numbers appearing in the problem, i.e. 3, 4 and 5 form a Pythagorean triple (i.e. $3^2 + 4^2 = 5^2$). This will be true more generally: we will see that $R^2 = a^2 + b^2$.

3. The expression $R\cos(x-\alpha)$.

We study the expression $R\cos(x-\alpha)$ and note that $\cos(x-\alpha)$ can be expanded using an addition formula.

$$\begin{aligned} R\cos(x-\alpha) &= R(\cos x\cos\alpha + \sin x\sin\alpha) \\ &= R\cos x\cos\alpha + R\sin x\sin\alpha \end{aligned}$$

We can re-order this expression as follows:

$$R\cos(x-\alpha) = (R\cos\alpha)\cos x + (R\sin\alpha)\sin x$$

So, if we want to write an expression of the form $a\cos x + b\sin x$ in the form $R\cos(x-\alpha)$ we can do this by comparing

$$a\cos x + b\sin x \qquad \text{with} \qquad (R\cos\alpha)\cos x + (R\sin\alpha)\sin x$$

Doing this we see that

$$a = R\cos\alpha \tag{1}$$
$$b = R\sin\alpha \tag{2}$$

How can we use these to find values for R and α ? By squaring each of Equations (1) and (2) and adding we find

$$\begin{aligned} a^2 + b^2 &= R^2\cos^2\alpha + R^2\sin^2\alpha \\ &= R^2(\cos^2\alpha + \sin^2\alpha) \\ &= R^2 \end{aligned}$$

since $\cos^2\alpha + \sin^2\alpha$ is always 1.

Hence

$$R = \sqrt{a^2 + b^2}$$

It is conventional to choose only the positive square root, and hence R will always be positive.

What about the α ?

We can find α by dividing Equation (2) by Equation (1) to give

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{b}{a}$$

so that

$$\tan\alpha = \frac{b}{a}$$

Knowing $\tan\alpha$ we can find α. So, now we can write any expression of the form $a\cos x + b\sin x$ in the form $R\cos(x-\alpha)$.

Key Point

$$a\cos x + b\sin x \quad \text{can be written as} \quad R\cos(x-\alpha)$$

where

$$R = \sqrt{a^2+b^2}, \qquad \tan\alpha = \frac{b}{a}$$

This is a very useful tool to have at one's disposal. It reduces the sum of two trigonometric functions to one trigonometric function. This can make it so much easier to deal with.

When using $\tan\alpha = \frac{b}{a}$ to find the value of α, care must always be taken to ensure that α lies in the correct quadrant. In particular, if either or both of a and b are negative you must be very careful. This will become apparent in the following examples.

In the next section we have a look at how we can use this result to solve equations. In the final section we use it to determine maxima and minima of functions.

Exercises 1

Each of the following expressions can be written in the form $R\cos(x-\alpha)$ with $-\pi < \alpha < \pi$. In each case determine the values of R and α (in radians) correct to 3 decimal places.

a)	$5\cos x + 12\sin x$	b)	$3\cos x + \sin x$	c)	$3\cos x - \sin x$	d)	$6\cos x + 5\sin x$
e)	$-5\cos x + 12\sin x$	f)	$4\cos x - \sin x$	g)	$-2\cos x - 3\sin x$	h)	$-\cos x + 3\sin x$
i)	$\cos x + \sin x$	j)	$\cos x - \sin x$	k)	$\sin x - \cos x$	l)	$-(\cos x + \sin x)$

4. Using the result to solve an equation

Example

Suppose we need to solve the trigonometric equation

$$\sqrt{2}\cos x + \sin x = 1$$

for values of x in the interval $-\pi < x < \pi$.

Comparing the left-hand side with the form $a\cos x + b\sin x$ we can identify a and b.

$$a = \sqrt{2} \qquad \text{and} \qquad b = 1$$

From the formula $R = \sqrt{a^2+b^2}$ we can calculate R:

$$R = \sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3}$$

Furthermore, recall that

$$R\cos\alpha = a \qquad \text{and} \qquad R\sin\alpha = b$$

so, with the known values of R, a and b,

$$\cos\alpha = \frac{\sqrt{2}}{\sqrt{3}} \qquad \text{and} \qquad \sin\alpha = \frac{1}{\sqrt{3}}$$

The fact that $\sin\alpha$ and $\cos\alpha$ (and therefore $\tan\alpha$) are all positive mean that α is an angle in the first quadrant. We can calculate it from either of the two previous equations or directly from

$$\tan\alpha = \frac{b}{a} \qquad \text{so that} \qquad \alpha = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{1}{\sqrt{2}} = 0.615 \text{ radians (3 d.p.)}$$

Therefore, the left-hand side of the given equation can be expressed in the form $\sqrt{3}\cos(x-0.615)$.

So the equation

$$\sqrt{2}\cos x + \sin x = 1$$

becomes

$$\sqrt{3}\cos(x-0.615) = 1$$

that is

$$\cos(x-0.615) = \frac{1}{\sqrt{3}}$$

This is very straightforward to solve. We seek the angle or angles which have a cosine of $\frac{1}{\sqrt{3}}$. Now if x lies in the interval $-\pi < x < \pi$ then $x - 0.615$ must lie in the interval

$$-\pi - 0.615 < x - 0.615 < \pi - 0.615$$

Figure 3 shows a graph of the cosine function over this interval. The angle on the right of the diagram which has a cosine of $\frac{1}{\sqrt{3}}$ can be found using a calculator and is 0.955. By symmetry the angle on the left is -0.955. Hence

$$\begin{aligned} x - 0.615 &= -0.955,\ 0.955 \\ x &= -0.955 + 0.615,\ 0.955 + 0.615 \\ &= -0.340,\ 1.570 \end{aligned}$$

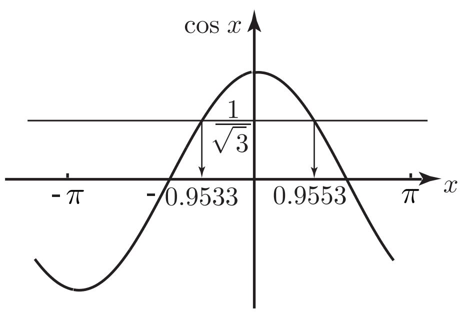

Figure 3. The cosine graph and a calculator enable us to find angles which have a cosine of $\frac{1}{\sqrt{3}}$.

Example

Suppose we wish to solve the equation

$$\cos x - \sqrt{3}\sin x = 2$$

for values of x in the interval $0° \leq x \leq 360°$.

We do this by first expressing the left-hand side in the form $R\cos(x-\alpha)$.

Comparing the left-hand side with the form $a\cos x + b\sin x$ we see that

$$a = 1 \qquad \text{and} \qquad b = -\sqrt{3}$$

Note in particular that b is negative, and this will be important when we calculate α. From the formula $R = \sqrt{a^2+b^2}$ we can calculate R:

$$R = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$$

Furthermore, recall that

$$R\cos\alpha = a \qquad \text{and} \qquad R\sin\alpha = b$$

so, with the known values of R, a and b,

$$\cos\alpha = \frac{1}{2} \qquad \text{and} \qquad \sin\alpha = \frac{-\sqrt{3}}{2}$$

The facts that $\sin\alpha$ is negative and $\cos\alpha$ is positive mean that α is an angle in the fourth quadrant. We can calculate it from either of the two previous equations or directly from

$$\tan\alpha = \frac{b}{a} \qquad \text{so that} \qquad \alpha = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{-\sqrt{3}}{1} = -60^\circ$$

Therefore, the left-hand side of the given equation can be expressed in the form $2\cos(x-(-60^\circ)) = 2\cos(x+60^\circ)$.

So the given equation becomes

$$2\cos(x+60^\circ) = 2$$

that is

$$\cos(x+60^\circ) = 1$$

We seek angles with a cosine equal to 1. Given that x lies in the interval $0^\circ < x < 360^\circ$ then $x + 60^\circ$ will lie in the interval

$$60^\circ < x + 60^\circ < 420^\circ$$

The only angle in this interval with cosine equal to 1 is 360°. It follows that

$$x + 60^\circ = 360^\circ$$

that is

$$x = 300^\circ$$

The only solution lying in the given interval is $x = 300^\circ$.

Exercises 2

Solve the following equations for $0 < x < 2\pi$

a) $2\cos x + \sin x = 1$	b) $2\cos x - \sin x = 1$	c) $-2\cos x - \sin x = 1$
d) $\cos x - 2\sin x = 1$	e) $\cos x + 2\sin x = 1$	f) $-\cos x + 2\sin x = 1$

5. Finding maximum and minimum values

We now consider an example involving maxima and minima.

Example

Consider the function $f(x) = 4\cos x + 3\sin x - 3$. We might be interested in a question such as 'what are its maximum and minimum values ?'.

From the earlier work in this unit we can express the first two terms as a single trigonometric function:

$$\begin{aligned} f(x) &= 4\cos x + 3\sin x - 3 \\ &= 5\cos(x-\alpha) - 3 \qquad \text{where} \qquad \tan\alpha = \frac{3}{4} \end{aligned}$$

Now the maximum value of the cosine function is 1 and this occurs when the angle $x - \alpha = 0$, i.e. when $x = \alpha$. So the maximum value of $f(x)$ must be $5 \times 1 - 3 = 2$.

What about the minimum value of $f(x)$? We know that the minimum value of the cosine function -1 and this occurs when $x - \alpha = \pi$, i.e. when $x = \pi + \alpha$. So the minimum value of $f(x)$ is $5 \times -1 - 3 = -8$.

So, very quickly and with the minimum amount of work we have established a maximum value and a minimum value, and based upon this information we could go ahead and sketch a graph of $f(x)$. We see that the form $R\cos(x-\alpha)$ is a very powerful form for us to know how to use and for us to be able to formulate from expressions such as $4\cos x + 3\sin x$.

Exercises 3

For each of the following functions determine the maximum value and the smallest positive angle (in radians, to three decimal places) at which the maximum value occurs.

a) $f(x) = 6 + 3\cos x + 4\sin x$ b) $f(x) = 3 - 4\cos x + 3\sin x$
c) $f(x) = 1 - 3\cos x - 4\sin x$ d) $f(x) = 2 + \cos x - \sin x$

Solutions

Exercises 1

a) 13, 1.176 b) $\sqrt{10}$, 0.322 c) $\sqrt{10}$, -0.322 d) $\sqrt{61}$, 0.695
e) 13, 1.966 f) $\sqrt{17}$, -0.245 g) $\sqrt{13}$, -2.159 h) $\sqrt{10}$, 1.893
i) $\sqrt{2}$, $\frac{\pi}{4}$ j) $\sqrt{2}$, $-\frac{\pi}{4}$ k) $\sqrt{2}$, $\frac{3\pi}{4}$ l) $\sqrt{2}$, $-\frac{3\pi}{4}$

Exercises 2

a) 1.571, $\left(\frac{\pi}{2}\right)$ or 5.640 b) 0.644 or 4.712 $\left(\frac{3\pi}{2}\right)$ c) 2.498 or 4.712 $\left(\frac{3\pi}{2}\right)$
d) 4.069 e) 2.214 f) 3.142 (π) or 0.927

Exercises 3

a) Max value 11 at 0.927 b) Max value 8 at 2.498
c) Max value 6 at 4.069 d) Max value $2 + \sqrt{2}$ at $\frac{7\pi}{4}$

