The scalar product

One of the ways in which two vectors can be combined is known as the **scalar product**. When we calculate the scalar product of two vectors the result, as the name suggests is a scalar, rather than a vector.

In this unit you will learn how to calculate the scalar product and meet some geometrical applications.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- define the scalar product of two vectors

- state some important properties of the scalar product

- calculate the scalar product when the two vectors are given in cartesian form

- use the scalar product in some geometrical applications

Contents

1. Introduction

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2. Definition of the scalar product

Study the two vectors $\mathbf{a}$ and $\mathbf{b}$ drawn in Figure 1. Note that we have drawn the two vectors so that their tails are at the same point. The angle between the two vectors has been labelled θ.
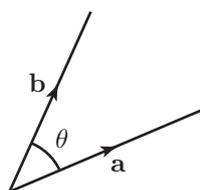

Figure 1. Two vectors, $\mathbf{a}$ and $\mathbf{b}$, drawn so that the angle between them is θ.

We define the scalar product of $\mathbf{a}$ and $\mathbf{b}$ as follows:

Key Point

The **scalar product** of $\mathbf{a}$ and $\mathbf{b}$ is defined to be

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta$$

where

$\quad$ $|\mathbf{a}|$ is the modulus, or magnitude of $\mathbf{a}$,

$\quad$ $|\mathbf{b}|$ is the modulus of $\mathbf{b}$, and

$\quad$ θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Note that the symbol for the scalar product is the dot $\cdot$, and so we sometimes refer to the scalar product as the dot product. Either name will do.

Example

Consider the two vectors **a** and **b** shown in Figure 2. Suppose **a** has modulus 4 units, **b** has modulus 5 units, and the angle between them is 60°, as shown.
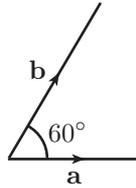

Figure 2. **a** and **b** have lengths 4 and 5 units respectively; the angle between them is 60°.

We can use the definition given above to find the scalar product of **a** and **b**.

$$
\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta \\
&= 4 \times 5 \times \cos 60° \\
&= 4 \times 5 \times \frac{1}{2} \\
&= 10
\end{aligned}
$$

So the scalar product of these vectors is the number 10. Note that the answer is a scalar, that is a number, rather than a vector. So, we have learnt a method of combining two vectors to produce a scalar.

3. Some properties of the scalar product

Commutativity and distributivity

Suppose for the two vectors in the previous example we calculate the product in a different order. That is, suppose we want to find **b** · **a**. The definition of **b** · **a** is

$$
\mathbf{b} \cdot \mathbf{a} = |\mathbf{b}|\,|\mathbf{a}|\,\cos\theta
$$

Performing the calculation using the numbers in the Example we find

$$
\begin{aligned}
\mathbf{b} \cdot \mathbf{a} &= |\mathbf{b}|\,|\mathbf{a}|\,\cos\theta \\
&= 5 \times 4 \times \cos 60° \\
&= 5 \times 4 \times \frac{1}{2} \\
&= 10
\end{aligned}
$$

So, we see that the result is the same which ever way around we perform the calculation. This is true in general:

$$
\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}
$$

This property of the scalar product is known as **commutativity**. We point it out because in another unit you can learn about another way of combining vectors known as the **vector product**. The vector product is not commutative so the order in which we write down the two vectors will be very important.

Key Point

The scalar product is **commutative**. This means

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

Another property of the scalar product is that it is **distributive over addition**. This means that

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

Although we shall not prove this result here we shall use it later on when we develop an alternative formula for finding the scalar product.

Key Point

The scalar product is **distributive over addition**. This means

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

and also, equivalently

$$(\mathbf{b} + \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{a}$$

The scalar product of two perpendicular vectors

Example

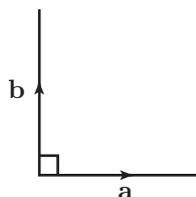Consider the two vectors $\mathbf{a}$ and $\mathbf{b}$ shown in Figure 3. The angle between them is $90°$, as shown.

Figure 3. The angle between $\mathbf{a}$ and $\mathbf{b}$ is $90°$.

We can use the definition to find the scalar product of **a** and **b**.

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}| \cos\theta \\ &= |\mathbf{a}|\,|\mathbf{b}| \cos 90° \\ &= 0 \end{aligned}$$

because $\cos 90° = 0$. This is true whatever the lengths of **a** and **b**. So the scalar product of two vectors which are at right-angles is always 0. We say that such vectors are **perpendicular** or **orthogonal**.

Key Point

For two perpendicular vectors

$$\mathbf{a} \cdot \mathbf{b} = 0$$

The converse of this statement is also true: if we have two non-zero vectors **a** and **b** and we find that their scalar product is zero, it follows that these vectors must be perpendicular. We can use this fact to test whether two vectors are perpendicular, as we shall see shortly.

4. The scalar product of two vectors given in cartesian form

We now consider how to find the scalar product of two vectors when these vectors are given in cartesian form, for example as

$$\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k} \qquad \text{and} \qquad \mathbf{b} = -5\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$$

where **i**, **j** and **k** are unit vectors in the directions of the x, y and z axes respectively.

First of all we need to develop a few results in the following examples.

Example

Suppose we want to find $\mathbf{i} \cdot \mathbf{j}$. These vectors are shown in Figure 4.

Note that because **i** and **j** lie along the x and y axes they must be perpendicular. So, from the result we have just established, the scalar product $\mathbf{i} \cdot \mathbf{j}$ must be zero. For the same reason, we

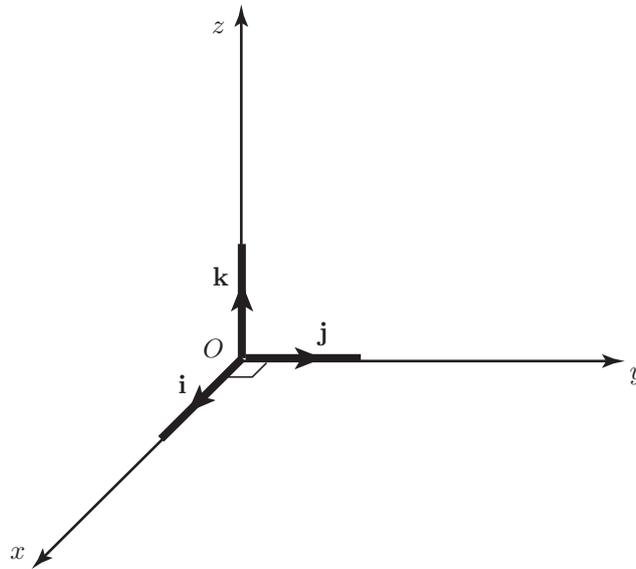obtain the same result if we calculate $\mathbf{i} \cdot \mathbf{k}$ and $\mathbf{j} \cdot \mathbf{k}$.

Figure 4. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are perpendicular.

Example

Suppose we want to find $\mathbf{i} \cdot \mathbf{i}$. Refer again to Figure 4. The vector $\mathbf{i}$ is a unit vector, so its length is 1 unit. The angle between a vector and itself must be zero. So

$$
\begin{aligned}
\mathbf{i} \cdot \mathbf{i} & = |\mathbf{i}|\,|\mathbf{i}|\,\cos 0° \\
& = 1 \times 1 \times 1 \\
& = 1
\end{aligned}
$$

since $\cos 0° = 1$.
For the same reason $\mathbf{j} \cdot \mathbf{j} = 1$ and $\mathbf{k} \cdot \mathbf{k} = 1$.

Key Point

If $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the directions of the x, y and z axes, then

$$\mathbf{i} \cdot \mathbf{j} = 0 \qquad \mathbf{i} \cdot \mathbf{k} = 0 \qquad \mathbf{j} \cdot \mathbf{k} = 0$$

$$\mathbf{i} \cdot \mathbf{i} = 1 \qquad \mathbf{j} \cdot \mathbf{j} = 1 \qquad \mathbf{k} \cdot \mathbf{k} = 1$$

We can use these results to develop a formula for finding the scalar product of two vectors given in cartesian form:

Suppose $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$
\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1\mathbf{i} \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&\quad + a_2\mathbf{j} \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&\quad\quad + a_3\mathbf{k} \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1\mathbf{i} \cdot b_1\mathbf{i} + a_1\mathbf{i} \cdot b_2\mathbf{j} + a_1\mathbf{i} \cdot b_3\mathbf{k} \\
&\quad + a_2\mathbf{j} \cdot b_1\mathbf{i} + a_2\mathbf{j} \cdot b_2\mathbf{j} + a_2\mathbf{j} \cdot b_3\mathbf{k} \\
&\quad\quad + a_3\mathbf{k} \cdot b_1\mathbf{i} + a_3\mathbf{k} \cdot b_2\mathbf{j} + a_3\mathbf{k} \cdot b_3\mathbf{k} \\
&= a_1 b_1 \mathbf{i} \cdot \mathbf{i} + a_1 b_2 \mathbf{i} \cdot \mathbf{j} + a_1 b_3 \mathbf{i} \cdot \mathbf{k} \\
&\quad + a_2 b_1 \mathbf{j} \cdot \mathbf{i} + a_2 b_2 \mathbf{j} \cdot \mathbf{j} + a_2 b_3 \mathbf{j} \cdot \mathbf{k} \\
&\quad\quad + a_3 b_1 \mathbf{k} \cdot \mathbf{i} + a_3 b_2 \mathbf{k} \cdot \mathbf{j} + a_3 b_3 \mathbf{k} \cdot \mathbf{k}
\end{aligned}
$$

Now from the previous Key Point most of these terms are zero. Those that are not simplify because $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$. We obtain

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

This is the formula which we can use to calculate a scalar product when we are given the cartesian components of the two vectors.

Key Point

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Note that a useful way to remember this is: multiply the $\mathbf{i}$ components together, multiply the $\mathbf{j}$ components together, multiply the $\mathbf{k}$ components together, and finally, add the results. On occasions you may see this form referred to as the **inner product** of the vectors $\mathbf{a}$ and $\mathbf{b}$. In the context of vectors this simply means the sum of the products of the corresponding vector components.

Example

Suppose we wish to find the scalar product of the two vectors $\mathbf{a} = 4\mathbf{i}+3\mathbf{j}+7\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i}+5\mathbf{j}+4\mathbf{k}$. The result we have just derived tells us to multiply the $\mathbf{i}$ components together, multiply the $\mathbf{j}$ components together, multiply the $\mathbf{k}$ components together, and finally add the results. So

$$
\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= (4)(2) + (3)(5) + (7)(4) \\
&= 8 + 15 + 28 \\
&= 51
\end{aligned}
$$

Example

Suppose we wish to find the scalar product of the two vectors $\mathbf{a} = -6\mathbf{i} + 3\mathbf{j} - 11\mathbf{k}$ and $\mathbf{b} = 12\mathbf{i} + 4\mathbf{k}$. Note that the $\mathbf{j}$ component of $\mathbf{b}$ is zero. So

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (-6)(12) + (3)(0) + (-11)(4) \\ &= -72 + 0 - 44 \\ &= -116 \end{aligned}$$

It is often useful to make use of column vector notation. Consider again the last example. Writing $\mathbf{a}$ and $\mathbf{b}$ as column vectors

$$\mathbf{a} = \begin{pmatrix} -6 \\ 3 \\ -11 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 12 \\ 0 \\ 4 \end{pmatrix}$$

the scalar product becomes

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} -6 \\ 3 \\ -11 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 0 \\ 4 \end{pmatrix} = (-6)(12) + (3)(0) + (-11)(4) = -116$$

Exercises 1.

1. If $\mathbf{a} = 4\mathbf{i} + 9\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j}$ find (a) $\mathbf{a} \cdot \mathbf{b}$, (b) $\mathbf{b} \cdot \mathbf{a}$, (c) $\mathbf{a} \cdot \mathbf{a}$, (d) $\mathbf{b} \cdot \mathbf{b}$.

2. Find the scalar product of the vectors $5\mathbf{i}$ and $8\mathbf{j}$.

3. If $\mathbf{p} = 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{q} = 2\mathbf{i} - \mathbf{j} + 11\mathbf{k}$ find (a) $\mathbf{p} \cdot \mathbf{q}$, (b) $\mathbf{q} \cdot \mathbf{p}$, (c) $\mathbf{p} \cdot \mathbf{p}$, (d) $\mathbf{q} \cdot \mathbf{q}$.

4. If $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$ show that $\mathbf{r} \cdot \mathbf{r} = |\mathbf{r}^2$.

5. If $\mathbf{a} = \begin{pmatrix} 2 \\ -3 \\ -4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 14 \\ 1 \end{pmatrix}$ find (a) $\mathbf{a} \cdot \mathbf{b}$, (b) $\mathbf{b} \cdot \mathbf{a}$, (c) $\mathbf{a} \cdot \mathbf{a}$, (d) $\mathbf{b} \cdot \mathbf{b}$.

6. Points A, B, and C have coordinates $(3,2,1)$, $(5,4,2)$, and $(-4, 2, 1)$ respectively. Find the scalar product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

5. Some applications of the scalar product

In this section we will look at some ways in which the scalar product can be used.

Using the scalar product to test whether two vectors are perpendicular

A common application of the scalar product is to test whether two vectors are perpendicular. From the definition

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta$$

If both the vectors $\mathbf{a}$ and $\mathbf{b}$ are non-zero and we find that $\mathbf{a} \cdot \mathbf{b} = 0$ then we can deduce $\cos\theta$ must be zero, so that $\theta = 90°$, i.e. $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. Consider the following example.

Example

Suppose we wish to test whether or not the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, where

$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

Note that neither $\mathbf{a}$ nor $\mathbf{b}$ is zero. Calculating the scalar product we find

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = (3)(1) + (2)(-2) + (-1)(-1) = 3 - 4 + 1 = 0$$

and so $\mathbf{a}$ and $\mathbf{b}$ are indeed perpendicular.

Key Point

If $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors for which $\mathbf{a} \cdot \mathbf{b} = 0$, then $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

Using the scalar product to find the angle between two vectors.

One of the common applications of the scalar product is to find the angle between two vectors when they are expressed in cartesian form.

From the definition of the scalar product

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta$$

We can rearrange this to obtain an expression for $\cos\theta$:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \tag{1}$$

If we are given $\mathbf{a}$ and $\mathbf{b}$ in cartesian form we can use the result obtained in Section 4 to calculate $\mathbf{a} \cdot \mathbf{b}$. We can also calculate the modulus of each of $\mathbf{a}$ and $\mathbf{b}$ since

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \qquad \text{and} \qquad |\mathbf{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

With this information Equation (1) can be used to find the angle between the two vectors.

Key Point

The angle, θ, between the two vectors $\mathbf{a}$ and $\mathbf{b}$ can be found from

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

Example

Suppose we wish to find the angle between the vectors $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$.

The scalar product, $\mathbf{a}\cdot\mathbf{b}$, has already been calculated on page 7 and found to be 51.

The modulus of each vector is found:

$$|\mathbf{a}| = \sqrt{4^2 + 3^2 + 7^2} = \sqrt{74}$$

$$|\mathbf{b}| = \sqrt{2^2 + 5^2 + 4^2} = \sqrt{45}$$

Then, from Equation (1),

$$
\begin{aligned}
\cos\theta &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}\\
&= \frac{51}{\sqrt{74}\sqrt{45}}\\
&= 0.8838 \text{ (4 sig fig)}
\end{aligned}
$$

Finally, using a calculator

$$\theta = \cos^{-1} 0.8838 = 27.90°$$

So the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is $27.90°$.

Finding the component of a vector in the direction of another vector

Another application of the scalar product is to find the component of one vector in the direction of another.

Consider Figure 5. We can think of the vector $\mathbf{b}$ being made up of a component in the direction of $\mathbf{a}$, $(\overrightarrow{OA})$, together with a perpendicular component, $(\overrightarrow{AB})$. The component in the direction of $\mathbf{a}$ is the **projection** of $\mathbf{b}$ onto $\mathbf{a}$.
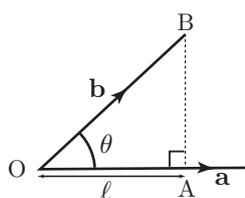
Figure 5. The projection of $\mathbf{b}$ onto $\mathbf{a}$ is ℓ.

From the right angled triangle OAB and using trigonometry we see that

$$\cos\theta = \frac{\ell}{|\mathbf{b}|}$$

and therefore

$$\ell = |\mathbf{b}|\cos\theta$$

Using the formula for $\cos\theta$ obtained in the previous application we have

$$
\begin{aligned}
\ell &= |\mathbf{b}|\cos\theta \\
&= |\mathbf{b}|\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \\
&= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}
\end{aligned}
$$

This can be written in the alternative form

$$\ell = \mathbf{b}\cdot\frac{\mathbf{a}}{|\mathbf{a}|}$$

So the projection of $\mathbf{b}$ onto $\mathbf{a}$ can be found by taking the scalar product of $\mathbf{b}$ and a unit vector in the direction of $\mathbf{a}$, i.e. $\ell = \mathbf{b}\cdot\hat{\mathbf{a}}$.

Key Point

The projection, ℓ, of $\mathbf{b}$ onto $\mathbf{a}$ can be found by taking the scalar product of $\mathbf{b}$ and a unit vector in the direction of $\mathbf{a}$:

$$\ell = \mathbf{b}\cdot\hat{\mathbf{a}}$$

Example

Suppose we wish to find the component of $\mathbf{b} = 3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ in the direction of $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$.
A unit vector in the direction of $\mathbf{a}$ is

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

Then

$$\mathbf{b}\cdot\hat{\mathbf{a}} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}}(3 - 1 + 4) = \frac{6}{\sqrt{3}} = 2\sqrt{3}.$$

So the component of $\mathbf{b} = 3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ in the direction of $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ is $2\sqrt{3}$.

Exercises 2.

1. Points A, B, and C have coordinates (3,2,1), (5,4,2), and $(-4, 2, 1)$ respectively. The scalar product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ has been found in the previous Exercises Q6. Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

2. Determine whether or not the vectors $2\mathbf{i} + 4\mathbf{j}$ and $-\mathbf{i} + \frac{1}{2}\mathbf{j}$ are perpendicular.

3. Evaluate $\mathbf{p} \cdot \mathbf{i}$ where $\mathbf{p} = 4\mathbf{i} + 8\mathbf{j}$. Hence find the angle that $\mathbf{p}$ makes with the x axis.

4. Find the component of the vector $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ in the direction of $\mathbf{a} = 7\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

Answers to Exercises

Exercises 1
1. (a) 30, (b) 30, (c) 97, (d) 13.
2. 0.
3. (a) 27, (b) 27, (c) 29, (d) 126.
4. Both equal 77.
5. (a) -50, (b) -50, (c) 29, (d) 201.
6. -14.

Exercises 2.
1. $131.8°$.
2. Their scalar product is zero. They are non-zero vectors. We deduce that they must be perpendicular.
3. $\mathbf{p} \cdot \mathbf{i} = 4$. The required angle is $63.4°$.
4. $\frac{15}{\sqrt{57}} = 1.987$ (3 d.p.).