Trigonometric Identities

mc-TY-trigids-2009-1

In this unit we are going to look at trigonometric identities and how to use them to solve trigonometric equations.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- derive three important identities
- use these identities in the solution of trigonometric equations

Contents

1. Introduction

In this unit we are going to look at trigonometric identities and how to use them to solve trigonometric equations. A trigonometric equation is an equation that involves a trigonometric function or functions. When we solve a trigonometric equation we find a value for the trigonometric function and then find the angle or angles that correspond to that particular trigonometric function.

2. Some important identities derived from a right-angled triangle

We begin our discussion with a right-angled triangle such as that shown in Figure 1.

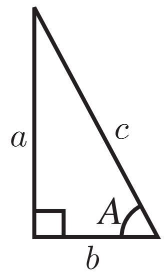

Figure 1. A right-angled triangle.

Using Pythagoras' theorem we know that

$$a^2 + b^2 = c^2$$

Dividing through by c^2 gives

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$

which we can rewrite as

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1 \qquad (1)$$

From Figure 1 we also observe that

$$\sin A = \frac{a}{c} \qquad \cos A = \frac{b}{c}$$

and so, from Equation (1),

$$(\sin A)^2 + (\cos A)^2 = 1$$

that is

$$\sin^2 A + \cos^2 A = 1 \qquad (2)$$

Note that $\sin^2 A$ is the commonly used notation for $(\sin A)^2$. Likewise, $\cos^2 A$ is the notation used for $(\cos A)^2$.

The mathematical expression in (2) is called an **identity** because it is true for **all** angles A, like this, in a right-angled triangle. However, we could have done this for the definitions of sine and cosine that come from a unit circle - in which case, this identity would be true for all angles A, no matter what their size.

This result is a very important trigonometric identity.

Key Point

$$\sin^2 A + \cos^2 A = 1$$

We want to develop this identity now to give us two more identities.

From

$$\sin^2 A + \cos^2 A = 1$$

we can divide through by $\cos^2 A$ to give

$$\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\cos^2 A} = \frac{1}{\cos^2 A}$$

But $\dfrac{\sin A}{\cos A} = \tan A$ and $\dfrac{1}{\cos A} = \sec A$. (Note that the definition of the secant of A is $\dfrac{1}{\cos A}$).
Hence

$$\tan^2 A + 1 = \sec^2 A$$

This is another important identity.

Key Point

$$\tan^2 A + 1 = \sec^2 A$$

Once again, returning to

$$\sin^2 A + \cos^2 A = 1$$

we can divide through by $\sin^2 A$ to give

$$\frac{\sin^2 A}{\sin^2 A} + \frac{\cos^2 A}{\sin^2 A} = \frac{1}{\sin^2 A}$$

But $\dfrac{\cos A}{\sin A} = \cot A$ and $\dfrac{1}{\sin A} = \mathrm{cosec} A$.
(Note that the definition of the cosecant of A is $\dfrac{1}{\sin A}$, and the definition of the cotangent of A is $\dfrac{1}{\tan A} = \dfrac{\cos A}{\sin A}$). Hence

$$1 + \cot^2 A = \mathrm{cosec}^2 A$$

Thus we have a third basic and fundamental identity.

Key Point

$$1 + \cot^2 A = \text{cosec}^2 A$$

3. Using the identities to solve equations

We are going to use these to help us solve particular kinds of trigonometrical equations.

Example

Suppose we wish to solve the equation

$$2\tan^2 x = \sec^2 x$$

for values of x in the interval $0 \leq x < 2\pi$.

We try to relate the given equation to one of our three identities.
We can use the identity $\sec^2 x = 1 + \tan^2 x$ to re-write the equation solely in terms of $\tan x$:

$$\begin{aligned} 2\tan^2 x &= \sec^2 x \\ 2\tan^2 x &= 1 + \tan^2 x \end{aligned}$$

from which

$$\tan^2 x = 1$$

Taking the square root then gives

$$\tan x = 1 \quad \text{or} \quad -1$$

The graph of $\tan x$ between 0 and 2π is shown in Figure 2. Note that the function values repeat every π radians.

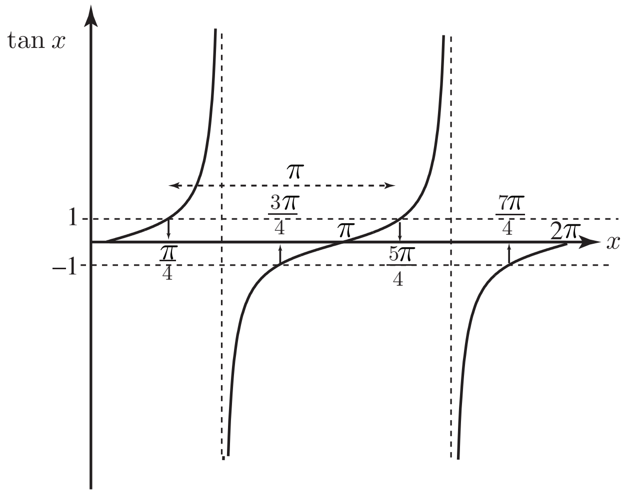

Figure 2. A graph of $\tan x$. The function values repeat every π radians.

A common result, which you should know (and learn if you don't), is that $\tan\frac{\pi}{4} = 1$. So $x = \frac{\pi}{4}$ is one solution of $\tan x = 1$. Inspection of the graph, and using its periodicity, yields the second solution, $x = \frac{5\pi}{4}$. We can also deduce solutions of $\tan x = -1$ from the same graph. These are $x = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$. So, altogether we have

$$x = \frac{\pi}{4},\ \frac{5\pi}{4},\ \frac{3\pi}{4},\ \frac{7\pi}{4}$$

These are the four solutions of the original equation.

Example

Suppose we wish to solve the equation

$$2\sin^2 x + \cos x = 1$$

for values of x in the interval $0 \le x < 2\pi$.

Using the identity $\sin^2 x + \cos^2 x = 1$ we can rewrite the equation in terms of $\cos x$. Instead of $\sin^2 x$ we can write $1 - \cos^2 x$. Then

$$\begin{aligned} 2\sin^2 x + \cos x &= 1 \\ 2(1 - \cos^2 x) + \cos x &= 1 \\ 2 - 2\cos^2 x + \cos x &= 1 \end{aligned}$$

This can be rearranged to

$$0 = 2\cos^2 x - \cos x - 1$$

This is a quadratic equation in $\cos x$ which can be factorised to

$$0 = (2\cos x + 1)(\cos x - 1)$$

Thus

$$2\cos x + 1 = 0 \qquad \text{or} \qquad \cos x - 1 = 0$$

from which

$$\cos x = -\frac{1}{2} \qquad \text{or} \qquad \cos x = 1$$

In order to determine the required values of x we consider the graph of $\cos x$ shown in Figure 3.

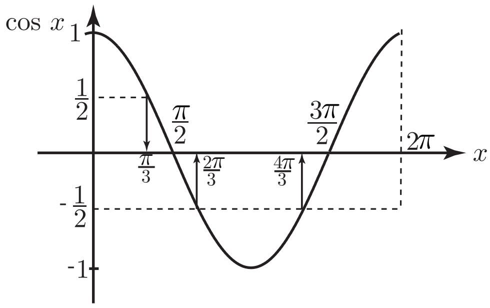

Figure 3. The graph of $\cos x$.

From the graph we see that $x = 0$ is a solution corresponding to that part of the equation $\cos x = 1$.

A well-known result is that $\cos\frac{\pi}{3} = \frac{1}{2}$. Using the symmetry of the graph we can deduce that the solutions corresponding to the equation $\cos x = -\frac{1}{2}$ are

$$x = \frac{2\pi}{3},\ \frac{4\pi}{3}$$

So the full solution is $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$.

Example

Suppose we wish to solve

$$3\cot^2 x = \operatorname{cosec} x - 1$$

for values of x in the interval $0 \le x < 2\pi$.

Here we use the identity $1 + \cot^2 x = \operatorname{cosec}^2 x$, and substitute for $\cot^2 x$.

$$\begin{aligned} 3\cot^2 x &= \operatorname{cosec} x - 1 \\ 3(\operatorname{cosec}^2 x - 1) &= \operatorname{cosec} x - 1 \\ 3\operatorname{cosec}^2 x - 3 &= \operatorname{cosec} x - 1 \end{aligned}$$

from which

$$3\operatorname{cosec}^2 x - \operatorname{cosec} x - 2 = 0$$

This is a quadratic equation in cosec x. It can be factorised as follows:

$$(3\operatorname{cosec} x + 2)(\operatorname{cosec} x - 1) = 0$$

and so

$$3\operatorname{cosec} x + 2 = 0 \qquad \text{or} \qquad \operatorname{cosec} x - 1 = 0$$

that is

$$\operatorname{cosec} x = -\frac{2}{3} \qquad \text{or} \qquad \operatorname{cosec} x = 1$$

This means

$$\frac{1}{\sin x} = -\frac{2}{3} \qquad \text{or} \qquad \frac{1}{\sin x} = 1$$

$$\sin x = -\frac{3}{2} \qquad \text{or} \qquad \sin x = 1$$

We sketch the graph of $\sin x$ over the interval $0 \le x < 2\pi$ as shown in Figure 4.

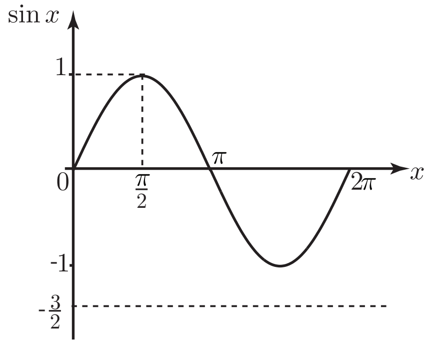

Figure 4. A graph of $\sin x$ showing that there are no values of x with sine equal to $-\frac{3}{2}$.

From the graph we deduce that there are no values of x satisfying $\sin x = -\frac{3}{2}$, and there is only one value satisfying $\sin x = 1$, namely $x = \dfrac{\pi}{2}$.

Example

Suppose we wish to solve

$$\cos x^2 - \sin^2 x = 0$$

This problem can be tackled in different ways. Here, we note that the left-hand side is the difference of two squares and so we can factorise it as

$$(\cos x - \sin x)(\cos x + \sin x) = 0$$

This means that

$$\cos x - \sin x = 0 \qquad \text{or} \qquad \cos x + \sin x = 0$$

so that

$$\sin x = \cos x \qquad \text{or} \qquad \sin x = -\cos x$$

Dividing through by $\cos x$ we obtain

$$\tan x = 1 \qquad \text{or} \qquad \tan x = -1$$

The solutions of these equations can be obtained by referring to the graph of $\tan x$ shown in Figure 5.

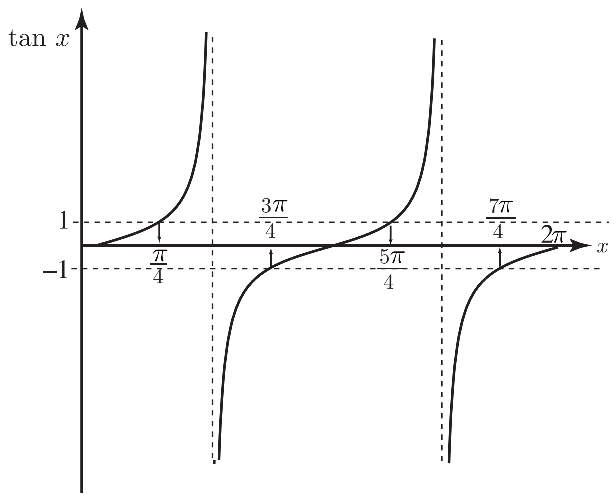

Figure 5. A graph of $\tan x$.

We deduce that the solutions are:

$$x = \frac{\pi}{4}, \; \frac{5\pi}{4}, \frac{3\pi}{4}, \; \frac{7\pi}{4}$$

So, by spotting that we could factorise the original equation we did not need to use an identity.

Let's have a look at the original equation again.

$$\cos^2 x - \sin^2 x = 0$$

This time we will make use of the identity $\sin^2 x = 1 - \cos^2 x$. Then

$$\cos^2 x - (1 - \cos^2 x) = 0$$

so that

$$\cos^2 x - 1 + \cos^2 x = 0$$

$$2\cos^2 x = 1$$

from which

$$\cos^2 x = \frac{1}{2}$$

Taking the square root

$$\cos x = \frac{1}{\sqrt{2}} \qquad \text{or} \qquad -\frac{1}{\sqrt{2}}$$

A graph of $\cos x$ will help to identify the required angles. Such a graph is shown in Figure 6. A standard result is that $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. From the graph we deduce the solutions:

$$x = \frac{\pi}{4}, \ \frac{7\pi}{4}, \ \frac{3\pi}{4}, \ \frac{5\pi}{4}$$

These are the same results as we found before using factorisation.

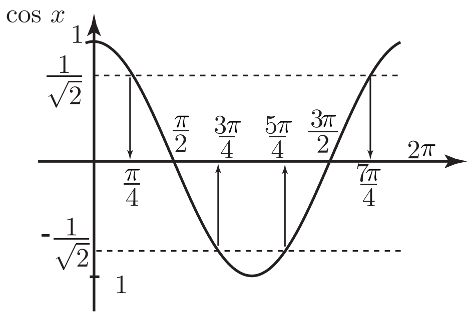

Figure 6. A graph of $\cos x$.

Exercises

1. In each question below you are given the quadrant an angle lies in and the value of one trigonometric ratio. In each case you should find the value of a stated second trigonometric ratio without finding the angle itself (i.e. without using inverse trigonometric functions). Give answers to 3 decimal places.

   (a) $0 < x < 90^o$, $\sin x = 0.3$, what is $\cos x$?

   (b) $0 < x < 90^o$, $\cos x = 0.6$, what is $\sin x$?

   (c) $90^o < x < 180^o$, $\tan x = -2$, what is $\sec x$?

   (d) $90^o < x < 180^o$, $\cot x = -0.5$, what is $\mathrm{cosec}\, x$?

   (e) $180^o < x < 270^o$, $\sec x = -\sqrt{10}$, what is $\tan x$?

   (f) $180^o < x < 270^o$, $\tan x = 4$, what is $\cos x$?

   (g) $270^o < x < 360^o$, $\cos x = 0.2$, what is $\cot x$?

   (h) $270^o < x < 360^o$, $\sec x = 2.5$, what is $\sin x$?

2. Solve the following trigonometric equations (state how many solutions as well as the values). Give answers to 2 decimal places (for degrees) or as fractions of π for radians.

   (a) $2\cos^2 x + 3\sin x = 3 \qquad 0 < x < 180^o$

   (b) $3\cos^2 x - \sin^2 x = 1 \qquad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

   (c) $1 + 17\tan x = 6\sec^2 x \qquad -90^o < x < 90^o$

   (d) $10\cot^2 x = \mathrm{cosec}\,^2 x \qquad 0 < x < 180^o$

   (e) $6\cos x - 5\sec x = \tan x \qquad -180^o < x < 180^o$

Answers

1.

a)	0.954	b)	0.8	c)	-2.236	d)	1.118
e)	3	f)	-0.243	g)	-0.204	h)	-0.917

2.

a) 3 solutions, 30^o, 90^o, 150^o

b) 2 solutions, $-\dfrac{\pi}{4}$, $\dfrac{\pi}{4}$

c) 2 solutions, 18.43^o, 68.20^o

d) 2 solutions, 71.57^o, 108.43^o

e) 4 solutions, -150^o, -30^o, 19.47^o, 160.53^o