Trigonometric ratios of an angle of any size

mc-TY-trigratiosanysize-2009-1

Knowledge of the trigonometrical ratios sine, cosine and tangent, is vital in very many fields of engineering, mathematics and physics. This unit explains how the sine, cosine and tangent of an arbitrarily sized angle can be found.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- define the ratios sine, cosine and tangent with reference to projections.
- use the trig ratios to solve problems involving triangles.

Contents

1. Introduction

In this session we are going to be looking at the definitions of sine, cosine and tangent for any size of angle. Let's first of all recall sine, cosine and tangent for angles in a right-angled triangle.

2. Trig ratios for angles in a right-angled triangle

Refer to the triangle in Figure 1.

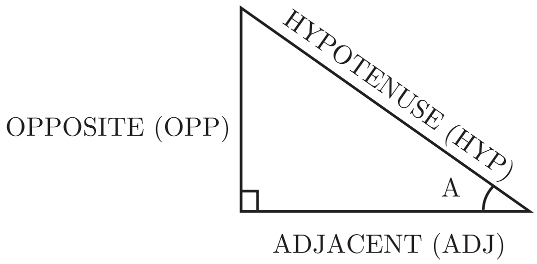

Figure 1. The side opposite the right-angle is called the hypotenuse

The side that is the longest side in the right-angled triangle and that is opposite the right angle is called the **hypotenuse**, or HYP for short.

The side that is opposite the angle A is called the **opposite** side, or OPP for short.

The side that runs alongside the angle A, and which is not the hypotenuse is called the **adjacent** side, or ADJ for short.

Recall the following important definitions:

Key Point

$$\sin A = \frac{\text{OPP}}{\text{HYP}}$$

$$\cos A = \frac{\text{ADJ}}{\text{HYP}}$$

$$\tan A = \frac{\text{OPP}}{\text{ADJ}}$$

However, these are defined only for acute angles, these are angles less than $90°$. What happens if we have an angle greater than $90°$, or less than $0°$? We explore this in the following section.

3. Angles

Consider Figure 2 which shows a circle of radius 1 unit centred at the origin. Imagine a point P on the circle which moves around the circle in an anticlockwise sense.

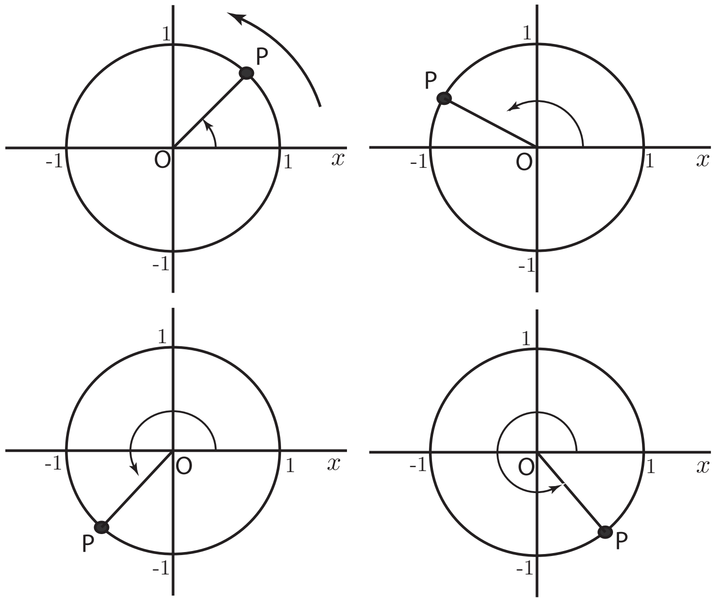

Figure 2. Angles measured anticlockwise from the positive x axis are deemed to be positive angles.

In the first diagram in Figure 2 the angle is acute, that is, it is greater than $0°$ but less than $90°$. When P moves into the second quadrant, the angle lies between $90°$ and $180°$. The angle is now obtuse. When P moves into the third quadrant, the angle is greater than $180°$ but less than $270°$. Finally in the fourth quadrant, the angle is greater than $270°$ but less that $360°$.

Consider now Figure 3. On these diagrams the arm OP is moving in a clockwise sense from the positive x axis. Such angles are conventionally taken to be **negative** angles.

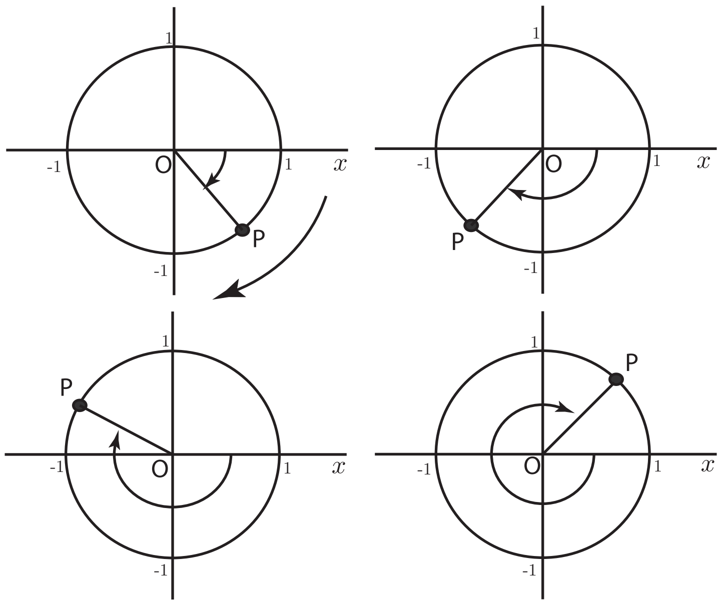

Figure 3. Angles measured clockwise from the positive x axis are deemed to be negative angles.

So, in this way we understand what is meant by an angle of any size, positive or negative. We now use these ideas together with our earlier definitions of sine, cosine and tangent in order to define these trig ratios for angles of any size.

4. The sine of an angle in any quadrant

Consider Figure 4 which shows a circle of radius 1 unit.

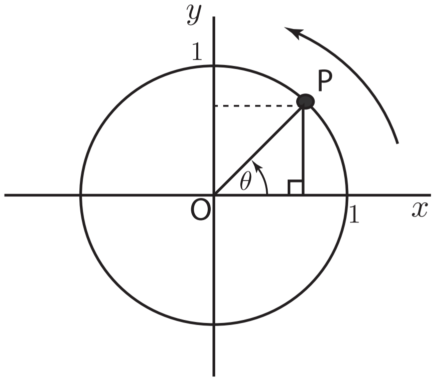

Figure 4. The side opposite θ has the same length as the projection of OP onto the y axis Oy.

The arm OP is in the first quadrant and we have dropped a perpendicular line down from P to the x axis in order to form the right-angled triangle shown.

Consider angle θ. The side opposite this angle has the same length as the projection of OP onto the y axis. So we define

$$\begin{aligned} \sin\theta &= \frac{\text{projection of } OP \text{ onto } Oy}{OP} \\ &= \text{projection of } OP \text{ onto } Oy \end{aligned}$$

since OP has length 1. This is entirely consistent with our earlier definition of $\sin\theta$ as $\frac{\text{OPP}}{\text{HYP}}$. Moreover, we can use this new definition to find the sine of any angle. Note that when the arm OP has rotated into the third and fourth quadrants the projection onto Oy will be negative.

Let's have a look at what that means in terms of a graph. Figure 5 shows the unit circle and the arm in various positions. The graph alongside is the projection of the arm onto the y axis. Corresponding points on both the circle and the graph are labelled A, B, C and so on. In other words this is the graph of $\sin\theta$.

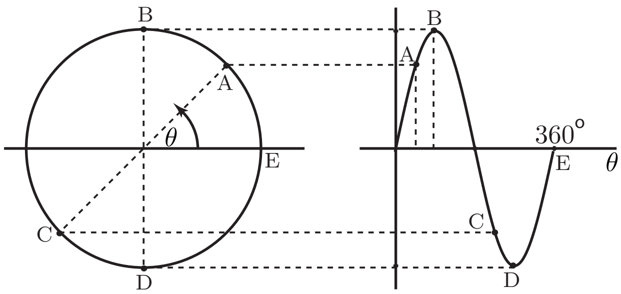

Figure 5. The graph of $\sin\theta$ can be drawn from the projections of the arm onto the y axis

We can produce a similar diagram for negative angles and we will obtain the graph shown in Figure 6. The whole pattern is reproduced every $360°$. In this way we can find the sine of any angle at all. Note also that the sine has a maximum value of 1, and a minimum value of -1. The graph never moves outside this range of values. To the left of $-360°$ and to the right of

$+360°$ the basic pattern simply repeats. This behaviour corresponds to arm OP moving around the circle again.

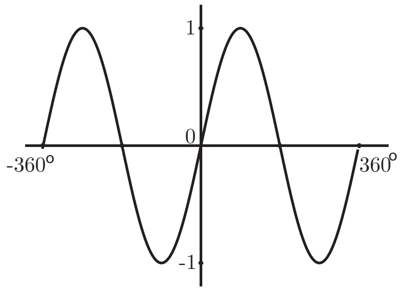

Figure 6. The graph of $\sin\theta$ extended to include negative angles

5. The cosine of any angle

To explore the cosine graph refer to Figure 7.

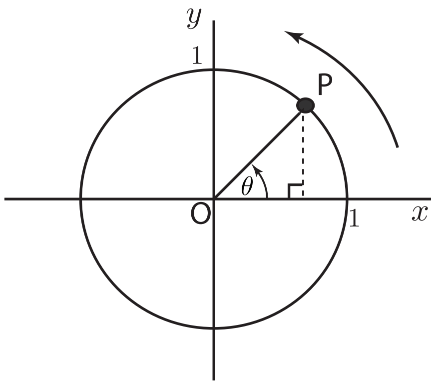

Figure 7. The side adjacent to angle θ has a length equal to the projection of OP onto the x axis.

We know that $\cos\theta = \dfrac{\text{ADJ}}{\text{HYP}}$. The length of the adjacent side is the same as the length of the projection of the arm OP onto the x axis. Hence we take the following as our definition of cosine:

$$\begin{aligned}\cos\theta &= \frac{\text{projection of } OP \text{ onto } Ox}{OP}\\ &= \text{projection of } OP \text{ onto } Ox\end{aligned}$$

since we are considering a unit circle and so $OP = 1$.

We can produce a graph as we did previously for $\sin\theta$ by finding the length of the projection of the arm OP onto the x axis. This is done by looking down on the arm from above as shown in Figure 8. For example, when $\theta = 0$, (point A), the projection has length 1. When $\theta = 90°$, the projection looks like a single point and has length zero (point B). When θ moves into the second and third quadrants, the x projection, and hence $\cos\theta$, is negative. In the fourth quadrant, the

x projection, and hence $\cos\theta$, is positive.

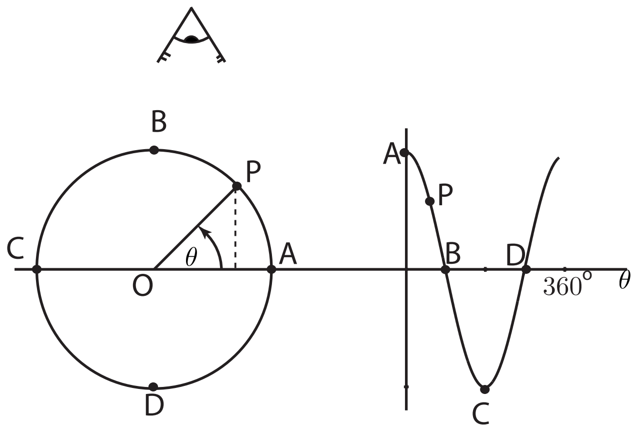

Figure 8. Look down on OP from above to find the projection of OP onto the x axis.

We can continue in this fashion to produce a cosine graph for negative angles. Doing so will result in the graph shown in Figure 9.

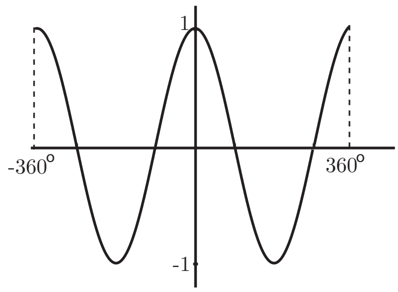

Figure 9. The graph of $\cos\theta$ extended to include negative angles.

This is a periodic graph. The same shape repeats every 360° as we move further to the left and to the right. Note also that the cosine has a maximum value of 1, and a minimum value of -1. The graph never moves outside this range of values.

Another important point to note is that the sine and cosine curves have the same shape. The cosine graph is the same as the sine except that it is displaced by 90°.

6. The tangent of any angle

Recall that tangent has already been defined as $\tan\theta = \dfrac{\text{OPP}}{\text{ADJ}}$.

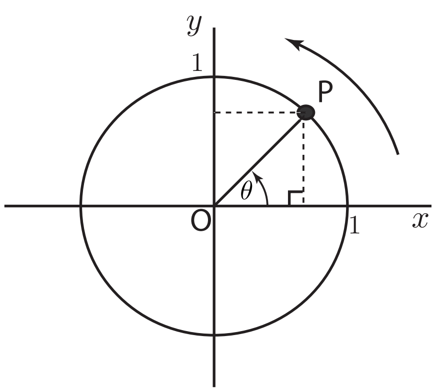

Figure 10. To find $\tan\theta$ we need to use projections onto both axes.

In terms of projections this definition becomes:

$$\tan\theta \quad = \quad \frac{\text{projection of } OP \text{ onto } Oy}{\text{projection of } OP \text{ onto } Ox}$$

An important result which follows immediately from comparing this definition with the earlier ones for sine and cosine is that:

$$\tan\theta \quad = \quad \frac{\sin\theta}{\cos\theta}$$

This gives us an identity which we need to learn and remember.

What does the graph of tangent look like? It's a little bit trickier to draw, but can be done by considering projections as outlined in Figure 11.

For example at point A, the projection onto Oy is zero, whilst the projection onto Ox is 1. Hence at A, $\tan\theta = \frac{0}{1} = 0$ and the corresponding point is indicated on the graph.
At point B both projections are equal and so $\tan\theta = 1$.
At points near to C the projection onto Oy is approaching 1, whilst the projection onto Ox is approaching zero. Hence the ratio

$$\tan\theta \quad = \quad \frac{\text{projection of } OP \text{ onto } Oy}{\text{projection of } OP \text{ onto } Ox}$$

becomes very large indeed. We indicate this by the dotted line on the graph. This line, called an **asymptote**, is approached by the graph as θ approaches $90°$.
Continuing in this fashion we can produce the graph of $\tan\theta$ for any angle θ, as shown in Figure 11.

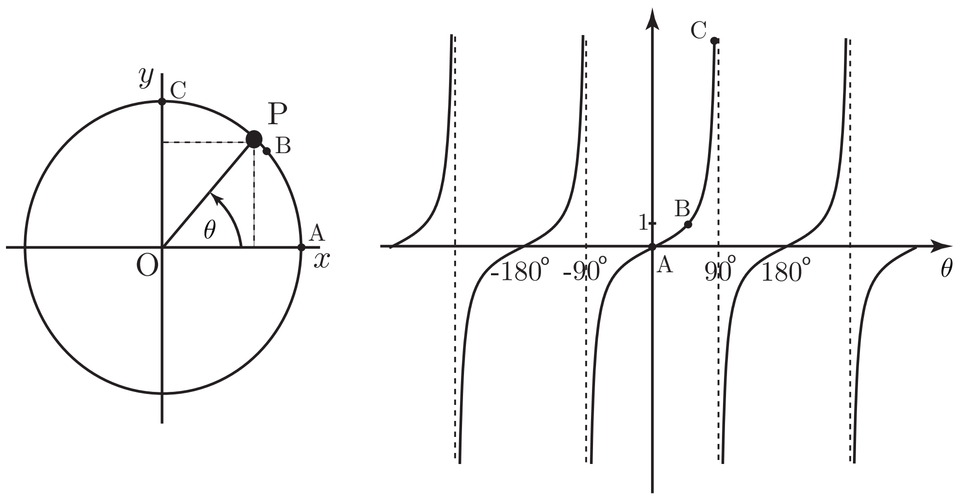

Figure 11. The graph of $\tan\theta$ can be found by considering projections

Note that the graph of $\tan\theta$ repeats every $180°$.

Exercise 1

Determine whether each of the following statements is true or false.

1. Sine is positive in the 1st and 4th quadrants.

2. The graph of cosine repeats itself every 180^o
3. The graph of tangent repeats itself every 180^o
4. Cosine is negative in the 2nd and 3rd quadrants
5. The graph of sine is continuous (i.e. has no breaks)
6. Tangent is negative in the 2nd and 4th quadrants
7. The graph of tangent is continuous (i.e. has no breaks)

Answers

Exercise 1

1. False 2. False 3. True 4. True 5. True 6. True 7. False

