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Differentiation for Economics and Business Studies Multi-Variable Functions

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This leaflet is an overview of the rules of partial differentiation and methods of optimization of functions in Economics and Business Studies.

Partial Differentiation

First partial derivatives: given a function $f(x, y)$:

- the first derivative of f with respect to x, $\frac{\partial f}{\partial x}$ or f_x, is obtained by differentiating f **treating** y **as a constant**;
- the first derivative of f with respect to y, $\frac{\partial f}{\partial y}$ or f_y, is obtained by differentiating f **treating** x **as a constant**;

There are four second partial derivatives:

- $\frac{\partial^2 f}{\partial x^2} = f_{xx}$ is obtained by differentiating $\frac{\partial f}{\partial x}$ with respect to x once more (treating y as a constant again);
- $\frac{\partial^2 f}{\partial y^2} = f_{yy}$ is obtained by differentiating $\frac{\partial f}{\partial y}$ with respect to y once more (treating x as a constant again);
- $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$ is obtained by differentiating $\frac{\partial f}{\partial x}$ with respect to y (now treating x as a constant);
- $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ is obtained by differentiating $\frac{\partial f}{\partial y}$ with respect to x (now treating y as a constant);

In most cases, and possibly in all cases you will encounter in your studies, we have: $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

Unconstrained Optimization

First Order Conditions (FOC): if a point (x_0, y_0) is such that $\frac{\partial f}{\partial x}(x_0, y_0) = 0$ **AND** $\frac{\partial f}{\partial y}(x_0, y_0) = 0$, then it is a stationary point.

Second Order Conditions (SOC): the stationary point (x_0, y_0) is:

- a minimum if $(\frac{\partial^2 f}{\partial x^2})(\frac{\partial^2 f}{\partial y^2}) - (\frac{\partial^2 f}{\partial x \partial y})^2 > 0$ and $\left(\frac{\partial^2 f}{\partial x^2} > 0, \quad \frac{\partial^2 f}{\partial y^2} > 0\right)$;
- a maximum if $(\frac{\partial^2 f}{\partial x^2})(\frac{\partial^2 f}{\partial y^2}) - (\frac{\partial^2 f}{\partial x \partial y})^2 > 0$ and $\left(\frac{\partial^2 f}{\partial x^2} < 0, \quad \frac{\partial^2 f}{\partial y^2} < 0\right)$;
- a saddle point if $(\frac{\partial^2 f}{\partial x^2})(\frac{\partial^2 f}{\partial y^2}) - (\frac{\partial^2 f}{\partial x \partial y})^2 < 0$.

Hessian matrix: this method is especially well-suited in the case of functions of more than two variables. The Hessian matrix of a function of n variables $f(x_1, x_2, \cdots, x_n)$ is the matrix with coefficients the second partial derivatives of f:

$$H(f) = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1} & \cdots & \cdots & f_{x_nx_n} \end{pmatrix}$$

The Hessian matrix is symmetric because we have $f_{x_ix_j} = f_{x_jx_i}$. We also define the principal minors of the Hessian matrix as the following determinants:

$$\left|f_{x_1x_1}\right|, \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{vmatrix},$$

$$\begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_2x_1} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} \end{vmatrix}, \cdots \det H(f)$$

Then we have:

- A stationary point (x_0, y_0) is a minimum if all the principal minors are strictly positive;
- A stationary point (x_0, y_0) is a maximum if the principal minors alternate sign, with $\left|f_{x_1x_1}\right| < 0$.

Constrained Optimization: Lagrange Multipliers

This method is used to find the optimum of a function of two or more variables when the variables are under a specific constraint:

Find the maximum (or minimum) of the function: $f(x_1, x_2, \cdots x_n)$ *given that*: $g(x_1, x_2, \cdots x_n) \leq M$

The constraint $g(x_1, x_2, \cdots x_n) \leq M$ will often have the form $p_1x_1 + p_2x_2 + \cdots + p_nx_n \leq M$. The Lagrangian multipliers method does not tell you whether the optimum is a maximum or a minimum; however, you will always be asked to locate either a minimum or a maximum, never to determine the nature of the optimum.

Given the Lagrangian function

$$L(x, y, \lambda) = f(x_1, x_2, \cdots x_n) + \lambda\Big(g(x_1, x_2, \cdots x_n) - M\Big)$$

the optimum of the function f is such that:

$$\frac{\partial L}{\partial x_1}(x_1, \cdots, x_n, \lambda) = 0$$
$$\vdots$$
$$\frac{\partial L}{\partial x_n}(x_1, \cdots, x_n, \lambda) = 0$$
$$\frac{\partial L}{\partial \lambda}(x_1, \cdots, x_n, \lambda) = 0$$

This gives a system of $n+1$ simultaneous equations and $n+1$ unknowns; solving the system will give you the value $(x_{10}, \cdots, x_{n0})$ of the optimum.